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NEW YORK UNIVERSITY
Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-172

Numerical Data Concerning the Effectiveness of Antennas

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P. Rabinowitz† and W. Magnus

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ABSTRACT

Numerical data are given for the eigenvalues of certain integral equations arising in antenna theory and discussed in an earlier report (EM-142). The integral equations are solved using approximations of various degree. The results are shown both in tables and in diagrams. A theoretical remark which can be used as a check-up is proved at the end of the report.

Introduction

This report evaluates certain results of Research Report EM-142 by N. Newman and W. Magnus. The problem considered here is to find the largest eigenvalue of each of the integral equations

$$(1) \qquad \lambda f(y) = \int_{-1}^{1} \gamma \ell \, \frac{J_{3/2}[k\ell(x - y)]}{[k\ell(x - y)]^{3/2}} \, f(x)dx$$

and

$$(2) \qquad \lambda^{*} f(y) = \int_{0}^{1} \frac{\gamma}{2} \ell \, \frac{J_{1/2}[k\ell(x - y)]}{[k\ell(x - y)]^{1/2}} \sqrt{1 - x^2} \, \sqrt{1 - y^2} \, f(x) \, dx$$

where $\gamma = 2(2\pi)^{3/2}k$. Each eigenvalue is to be found as a function of the parameter $k\ell$. The range of the parameter studied will be $0 < k\ell \leq 10$. The eigenvalue λ represents the ratio of the radiated energy $\mathcal{E}$ of an antenna to its heat loss $\mathcal{H}$, while λ^{*} represents the ratio of the energy $\mathcal{E}^{*}$ radiated into a half space to the total heat loss $\mathcal{H}$, in both cases k being the wave number and 2ℓ the length of the antenna.

In Research Report EM-142 this ratio was called the gain of the antenna. We are indebted to Dr. W. Gerbes for calling our attention to the fact that this is a misleading terminology, since the term "gain" has an established and different usage in antenna theory. An expression like "effectiveness" or "degree of efficiency" for the ratio of radiated energy versus heat losses would seem to be more appropriate.

I. Method of Solution and Results

A method of successive polynomial approximation is used on both integral equations.

A. Equation 1.

In equation (1) the kernel can be expanded in a double Taylor series:

$$\frac{J_{3/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{3/2}} = \left(\frac{1}{2}\right)^{3/2}\left[\frac{1}{\Gamma(5/2)} - \frac{\left[k\ell(x-y)\right]^2}{1!2^2\,\Gamma(7/2)} + \frac{\left[k\ell(x-y)\right]^4}{2!2^4\,\Gamma(9/2)} - \frac{\left[k\ell(x-y)\right]^6}{3!2^6\,\Gamma(11/2)}\right]$$

f(x) and f(y) can be expanded in the Taylor series:

$$f(x) = \Sigma f_n x^n \quad f(y) = \Sigma f_n y^n$$

As a first approximation, one term of the expansion of the kernel will be taken and in each succeeding approximation another term will be added. In each approximation a sufficient number of terms from the expansions of f(x) and f(y) is employed to make the resulting equation an identity in y. Eliminating x by integration, and equating coefficients of like powers of y on each side of the equation, a system of linear equations is obtained in which the unknowns appear as f_0, f_1, etc., and λ is an eigenvalue of the coefficient matrix.

Four approximations are made. The results are the following:

First Approximation:

$$\lambda f_0 = \frac{\gamma\ell}{\sqrt{2}\Gamma(5/2)} f_0$$

Second Approximation:

We shall write P for $k\ell$ and t for $\gamma\ell/\sqrt{2}$. We have:

Letting $k\ell = P$ and $\frac{\gamma\ell}{2} = t$

$$\lambda f_0 = t\left\{\left[\frac{1}{\Gamma(5/2)} - \frac{P^2}{12\Gamma(7/2)}\right] f_0 + \left[\frac{1}{3\Gamma(5/2)} - \frac{P^2}{20\Gamma(7/2)}\right] f_2\right\}$$

$$\lambda f_1 = \frac{tP^2}{6\Gamma(7/2)} f_1$$

$$\lambda f_2 = t\left\{-\frac{P^2}{4\Gamma(7/2)} f_0 - \frac{P^2}{12\Gamma(7/2)} f_2\right\}$$

Third Approximation:

$$\lambda f_0 = t\left\{\left[\frac{1}{\Gamma(5/2)} - \frac{P^2}{12\Gamma(7/2)} + \frac{P^4}{160\Gamma(9/2)}\right] f_0\right.$$
$$+ \left[\frac{1}{3\Gamma(5/2)} - \frac{P^2}{20\Gamma(7/2)} + \frac{P^4}{224\Gamma(9/2)}\right] f_2$$
$$\left.+ \left[\frac{1}{5\Gamma(5/2)} - \frac{P^2}{28\Gamma(7/2)} + \frac{P^4}{288\Gamma(9/2)}\right] f_4\right\}$$

$$\lambda f_1 = t\left[\frac{P^2}{6\Gamma(7/2)} - \frac{P^4}{40\Gamma(9/2)}\right] f_1 + \left[\frac{P^2}{10\Gamma(7/2)} - \frac{P^4}{56\Gamma(9/2)}\right] f_3$$

$$\lambda f_2 = t\left[-\frac{P^2}{4\Gamma(7/2)} + \frac{P^4}{16\Gamma(9/2)}\right] f_0 + \left[-\frac{P^2}{12\Gamma(7/2)} + \frac{3P^4}{80\Gamma(9/2)}\right] f_2$$
$$+ \left[-\frac{P^2}{20\Gamma(7/2)} + \frac{3P^4}{112\Gamma(9/2)}\right] f_4$$

$$\lambda f_3 = t\left\{-\frac{P^4}{24\Gamma(9/2)} f_1 - \frac{P^4}{40\Gamma(9/2)} f_3\right\}$$

$$\lambda f_4 = t\left\{\frac{P^4}{32\Gamma(9/2)} f_0 + \frac{P^4}{96\Gamma(9/2)} f_2 + \frac{P^4}{160\Gamma(9/2)} f_4\right\}$$

Fourth Approximation:

$$\lambda f_0 = t\left\{\left[\frac{1}{\Gamma(5/2)} - \frac{P^2}{12\Gamma(7/2)} + \frac{P^4}{160\Gamma(9/2)} - \frac{P^6}{2688\Gamma(11/2)}\right] f_0\right.$$
$$+ \left[\frac{1}{3\Gamma(5/2)} - \frac{P^2}{20\Gamma(7/2)} + \frac{P^4}{224\Gamma(9/2)} - \frac{P^6}{3456\Gamma(11/2)}\right] f_2$$

$$+ \left[\frac{1}{5\Gamma(5/2)} - \frac{P^2}{28\Gamma(7/2)} + \frac{P^4}{288\Gamma(9/2)} - \frac{P^6}{4224\Gamma(11/2)}\right] f_4$$

$$\left. + \left[\frac{1}{7\Gamma(5/2)} - \frac{P^2}{36\Gamma(7/2)} + \frac{P^4}{352\Gamma(9/2)} - \frac{P^6}{4992\Gamma(11/2)}\right] f_6 \right\}$$

$$\lambda f_1 = t \left\{ \left[\frac{P^2}{6\Gamma(7/2)} - \frac{P^4}{40\Gamma(9/2)} + \frac{P^6}{448(11/2)}\right] f_1 \right.$$

$$+ \left[\frac{P^2}{10\Gamma(7/2)} - \frac{P^4}{56\Gamma(9/2)} + \frac{P^6}{576\Gamma(11/2)}\right] f_3$$

$$\left. + \left[\frac{P^2}{14\Gamma(7/2)} - \frac{P^4}{72\Gamma(9/2)} + \frac{P^6}{704\Gamma(11/2)}\right] f_5 \right\}$$

$$\lambda f_2 = t \left\{ \left[- \frac{P^2}{4\Gamma(7/2)} + \frac{P^4}{16\Gamma(9/2)} - \frac{P^6}{128\Gamma(11/2)}\right] f_0 \right.$$

$$+ \left[- \frac{P^3}{12\Gamma(7/2)} + \frac{3P^4}{80\Gamma(9/2)} - \frac{5P^6}{896\Gamma(11/2)}\right] f_2$$

$$+ \left[- \frac{P^2}{20\Gamma(7/2)} + \frac{3P^4}{112\Gamma(9/2)} - \frac{5P^6}{1152\Gamma(11/2)}\right] f_4$$

$$+ \left[- \frac{P^2}{28\Gamma(7/2)} + \frac{P^4}{48\Gamma(9/2)} - \frac{5P^6}{1408\Gamma(11/2)}\right]$$

$$\lambda f_3 = t \left\{ \left[- \frac{P^4}{24\Gamma(9/2)} + \frac{P^6}{96\Gamma(11/2)}\right] f_1 + \left[\frac{-P^4}{40\Gamma(9/2)} + \frac{5P^6}{672\Gamma(11/2)}\right] f_3 \right.$$

$$\left. + \left[-\frac{P^4}{56\Gamma(9/2)} + \frac{5P^6}{864\Gamma(11/2)}\right] f_5 \right\}$$

$$\lambda f_4 = t \left\{ \left[\frac{P^4}{32\Gamma(9/2)} - \frac{5P^6}{384\Gamma(11/2)}\right] f_0 + \left[\frac{P^4}{96\Gamma(9/2)} - \frac{P^6}{128\Gamma(11/2)}\right] f_2 \right.$$

$$\left. + \left[\frac{P^4}{160\Gamma(9/2)} - \frac{5P^6}{896\Gamma(11/2)}\right] f_4 + \left[\frac{P^4}{224\Gamma(9/2)} - \frac{5P^6}{1152\Gamma(11/2)}\right] f_6 \right\}$$

$$\lambda f_5 = t \left\{ \frac{P^6}{192\Gamma(11/2)} f_1 + \frac{P^6}{320\Gamma(11/2)} f_3 + \frac{P^6}{448\Gamma(11/2)} f_5 \right\}$$

$$\lambda f_6 = t \left\{ \frac{-P^6}{384\Gamma(11/2)} f_0 - \frac{P^6}{1152\Gamma(11/2)} f_2 - \frac{P^6}{1920\Gamma(11/2)} f_4 - \frac{P^6}{2688\Gamma(11/2)} f_6 \right\}$$

To obtain numerical results, an (IBM-704) digital computer was used. The eigenvalues of the matrices were found for the interval $0 < P \leq 10$. The positive eigenvalues were tabulated and graphed. We include these results here.

In the following tables, the first column P denotes the value of the parameter and λ_1, λ_2, etc. denote the positive eigenvalues corresponding to P.

Equation 1 First Approximation

P	λ_1	P	λ_1
0.1	1.710	3.5	59.854
0.2	3.420	4.0	68.405
0.3	5.131	4.5	76.955
0.4	6.841	5.0	85.506
0.5	8.551	5.5	94.057
0.6	10.261	6.0	102.608
0.7	11.971	6.5	111.158
0.8	13.681	7.0	119.709
0.9	15.391	7.5	128.260
1.0	17.101	8.0	136.810
1.5	25.652	8.5	145.361
2.0	34.203	9.0	153.912
2.5	42.753	9.5	162.191
3.0	51.304	10.0	171.013

Equation 1 Second Approximation

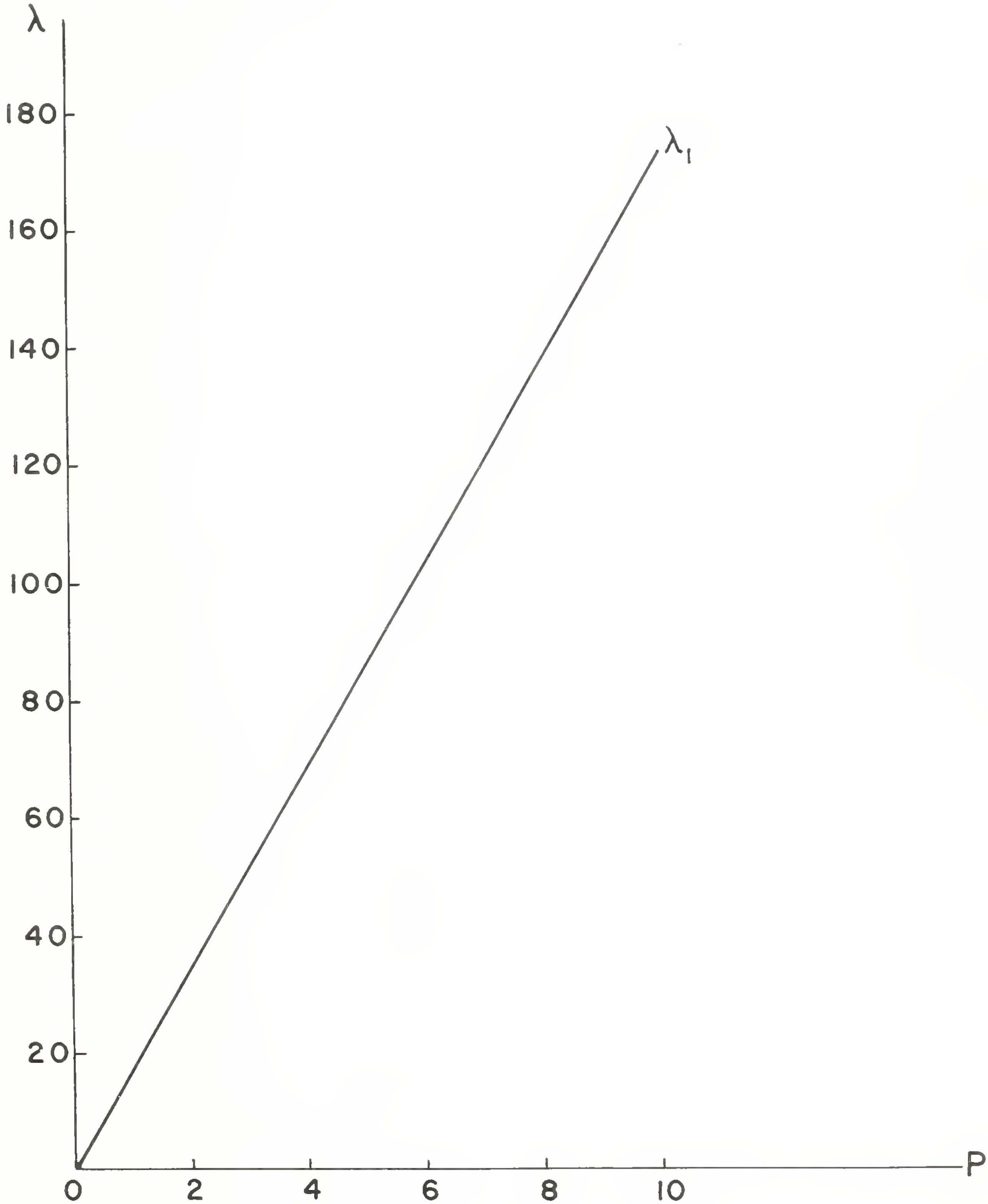
P	λ_1	λ_2	P	λ_1	λ_2
0.1	1.7090	0.0011	3.5	28.0256	48.891
0.2	3.4111	0.0091	4.0	30.4306	72.985
0.3	5.0996	0.0307	4.5	34.9059	103.920
0.4	6.7677	0.0727	5.0	41.3125	142.100
0.5	8.4086	0.1421	5.5	49.5608	189.688
0.6	10.0157	0.2455	6.0	59.6543	245.500
0.7	11.5825	0.3898	6.5	71.6562	313.262
0.8	13.1025	0.5843	7.0	85.6632	389.800
0.9	14.5696	0.8320	7.5	101.7905	481.094
1.0	15.9775	1.140	8.0	120.1640	584.800
1.5	21.9393	3.847	8.5	140.9163	700.350
2.0	25.7290	9.123	9.0	164.1840	832.000
2.5	27.2678	17.816	9.5	190.1064	976.309
3.0	27.4312	30.791	10.0	218.8245	1140.400

Equation 1 Third Approximation

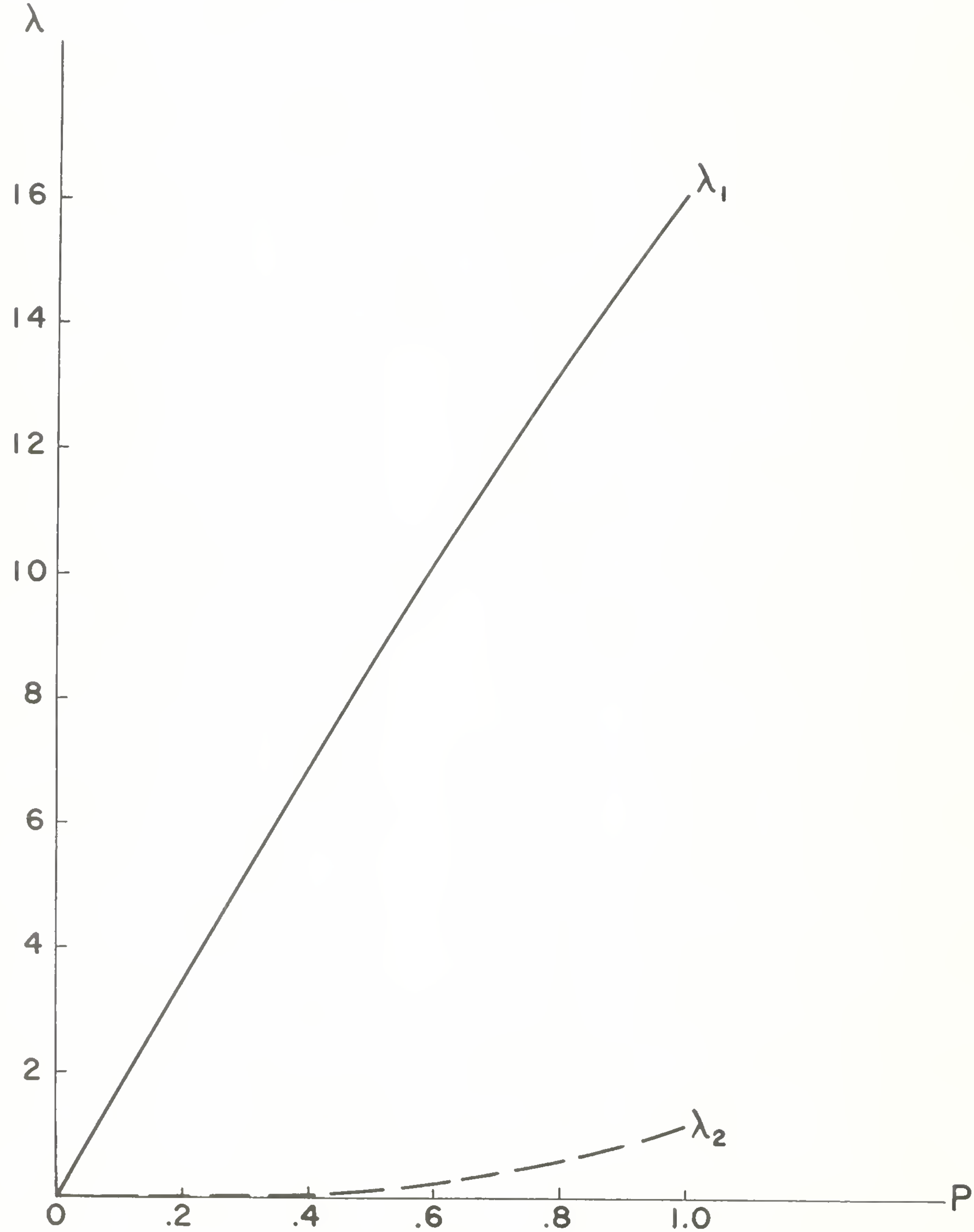
P	λ_1	λ_2	λ_3
0.1	1.7090	0.0011	0.0000
0.2	3.4112	0.0091	0.0000
0.3	5.0998	0.0305	0.0000
0.4	6.7684	0.0720	0.0002
0.5	8.4106	0.1394	0.0006
0.6	10.0207	0.2387	0.0014
0.7	11.5932	0.3746	0.0031
0.8	13.1232	0.5517	0.0061
0.9	14.6065	0.7736	0.0113
1.0	16.0395	1.0427	0.0195
1.5	22.3787	3.1137	0.1681
2.0	27.3896	6.0689	0.8218
2.5	31.6035	8.7622	2.8899
3.0	36.4377	9.7758	7.8738
3.5	46.3560	10.0400	16.0965
4.0	73.1995	12.1460	22.7928
4.5	129.3748	16.6362	25.4182
5.0	226.2173	23.7918	25.9281
5.5	379.3885	34.2454	26.0524
6.0	609.1100	48.8882	27.3007
6.5	940.0600	68.8303	30.5615
7.0	1401.6548	95.3972	36.1030
7.5	2028.3870	130.1384	44.0668
8.0	2860.1969	174.8410	54.6938
8.5	3942.8534	231.5443	68.3456
9.0	5328.3416	302.5562	85.4862
9.5	7075.2467	390.4691	106.6683
10.0	9249.1440	498.1762	132.5267

Equation 1 Fourth Approximation

P	λ_1	λ_2	λ_3	λ_4
0.1	1.7090	0.0011	0.0000	0.0000
0.2	3.4112	0.0091	0.0000	0.0000
0.3	5.0998	0.0305	0.0000	0.0000
0.4	6.7684	0.0720	0.0002	0.0000
0.5	8.4106	0.1395	0.0005	0.0000
0.6	10.0206	0.2388	0.0013	0.0000
0.7	11.5930	0.3750	0.0029	0.0000
0.8	13.1227	0.5527	0.0056	0.0000
0.9	14.6053	0.7758	0.0100	0.0001
1.0	16.0371	1.0473	0.0167	0.0002
1.5	22.3399	3.1901	0.1184	0.0035
2.0	27.1247	6.6181	0.4272	0.0357
2.5	30.4378	11.1915	0.9397	0.2313
3.0	32.3409	17.2900	1.2038	1.0485
3.5	32.9198	27.5206	1.3851	3.3083
4.0	33.0194	50.8453	2.3260	6.8545
4.5	34.0279	107.4889	4.4039	9.7448
5.0	36.6575	232.1929	8.2137	10.8596
5.5	42.6341	481.4483	13.8524	10.9634
6.0	57.3161	943.4997	19.2576	12.0415
6.5	88.3405	1749.7626	22.0188	15.0464
7.0	142.9913	3088.4077	22.6746	20.2638
7.5	231.0920	5220.4828	22.7624	28.2006
8.0	366.6423	8498.7921	23.8321	39.6743
8.5	568.4678	13389.7327	26.8703	55.8090
9.0	861.0733	20498.3076	32.2569	78.0608
9.5	1275.7182	30596.5022	40.2812	108.2665
10.0	1851.6699	44655.2603	51.3914	148.7058



λ: First Approximation



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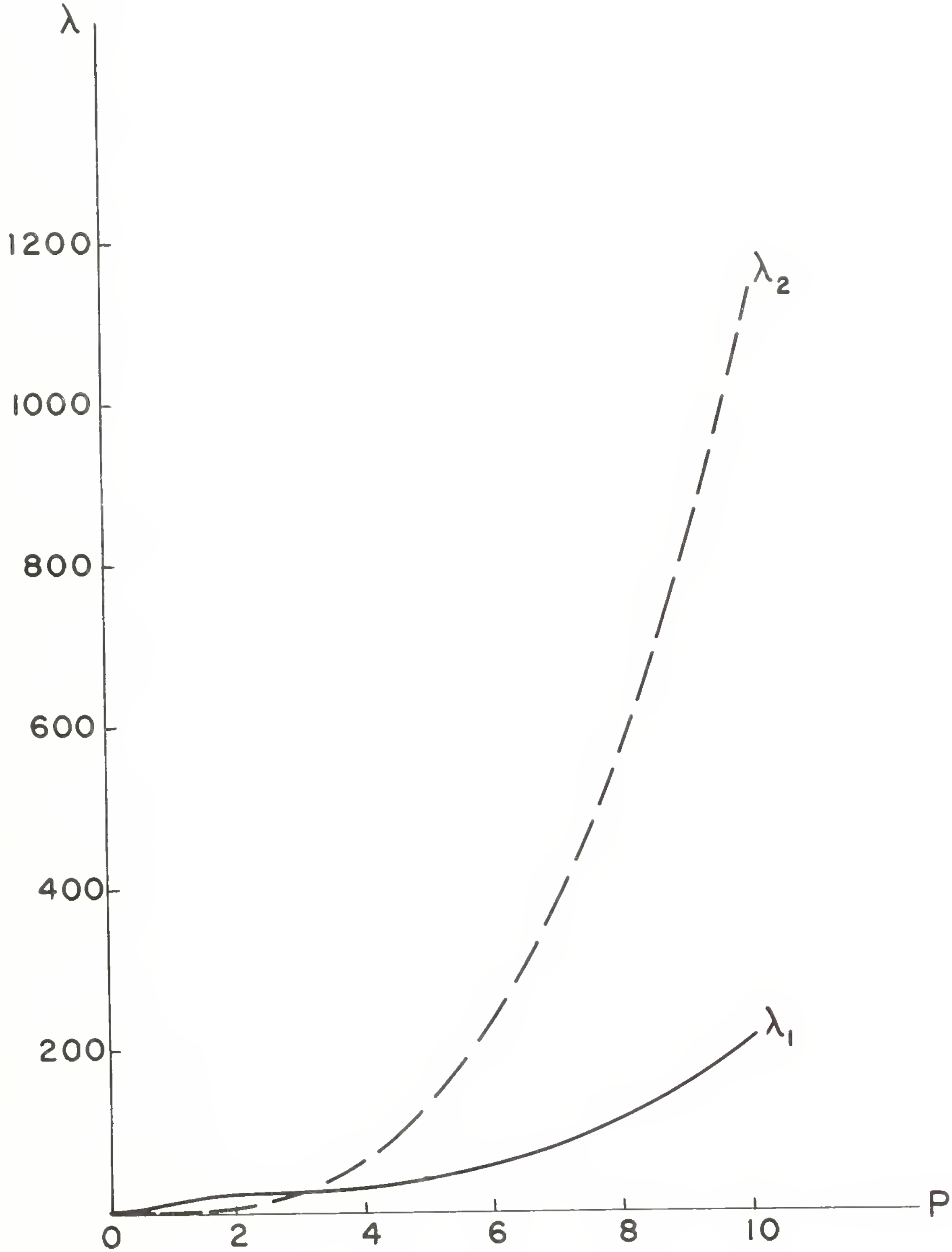

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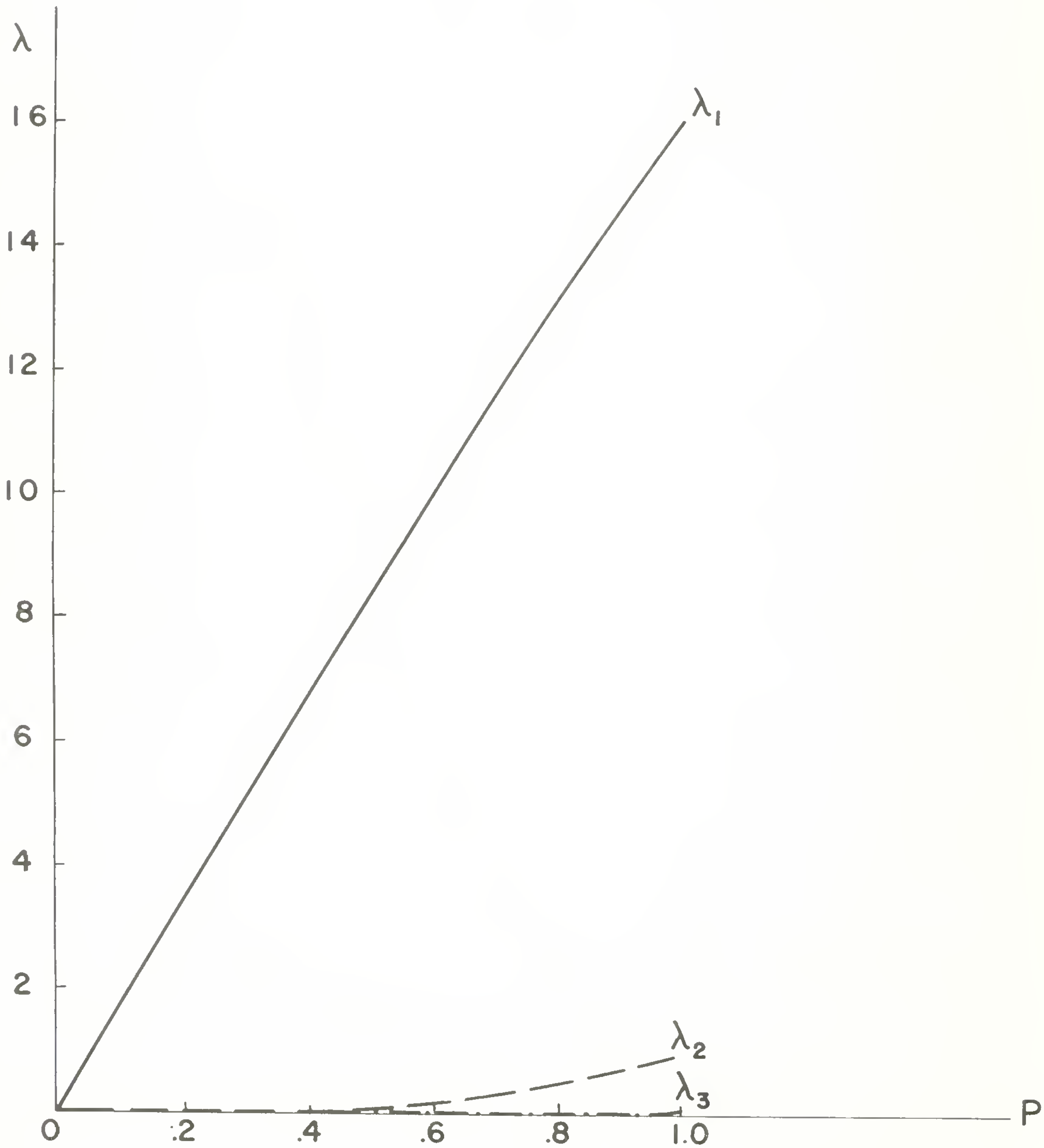
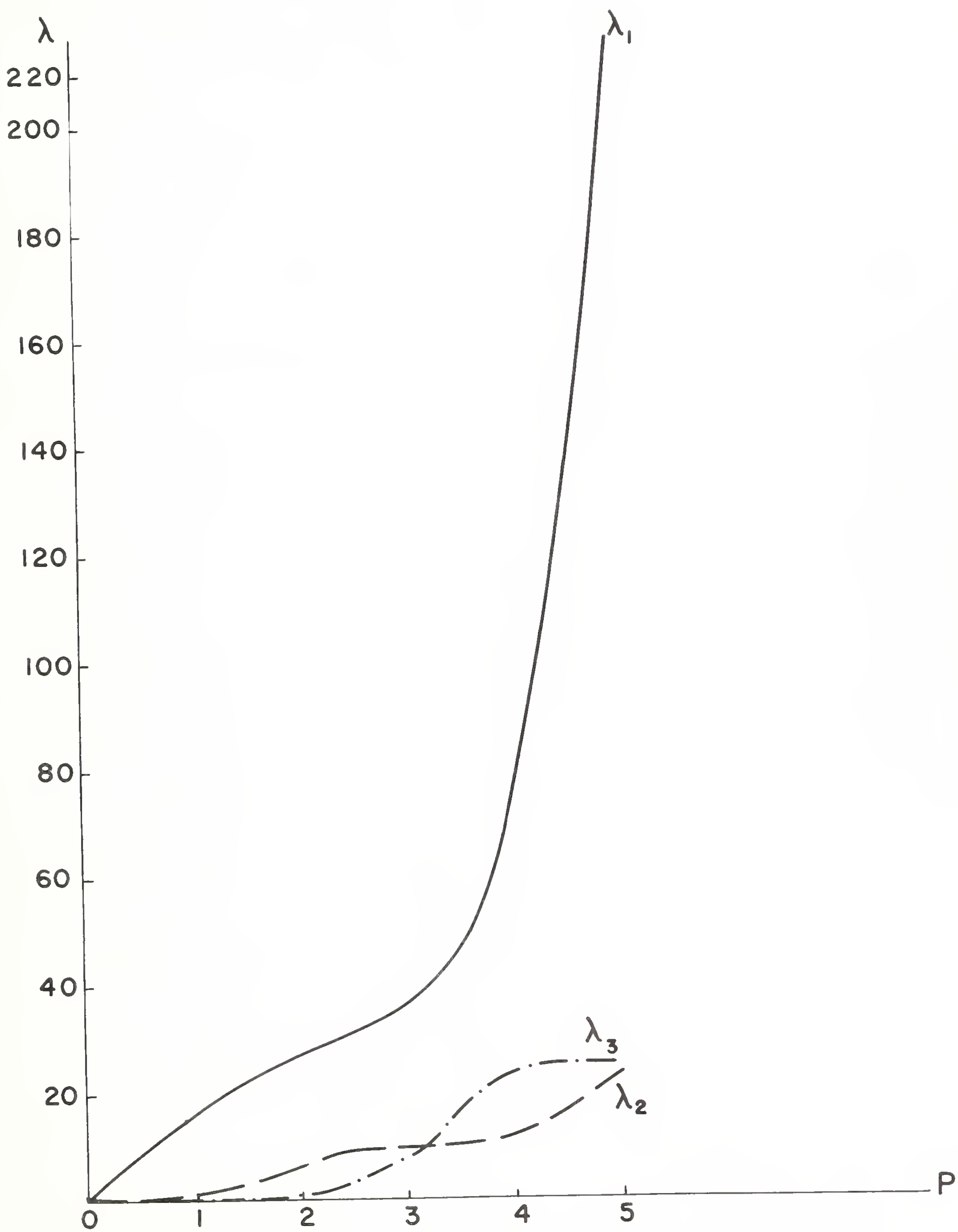


λ: Third Approximation



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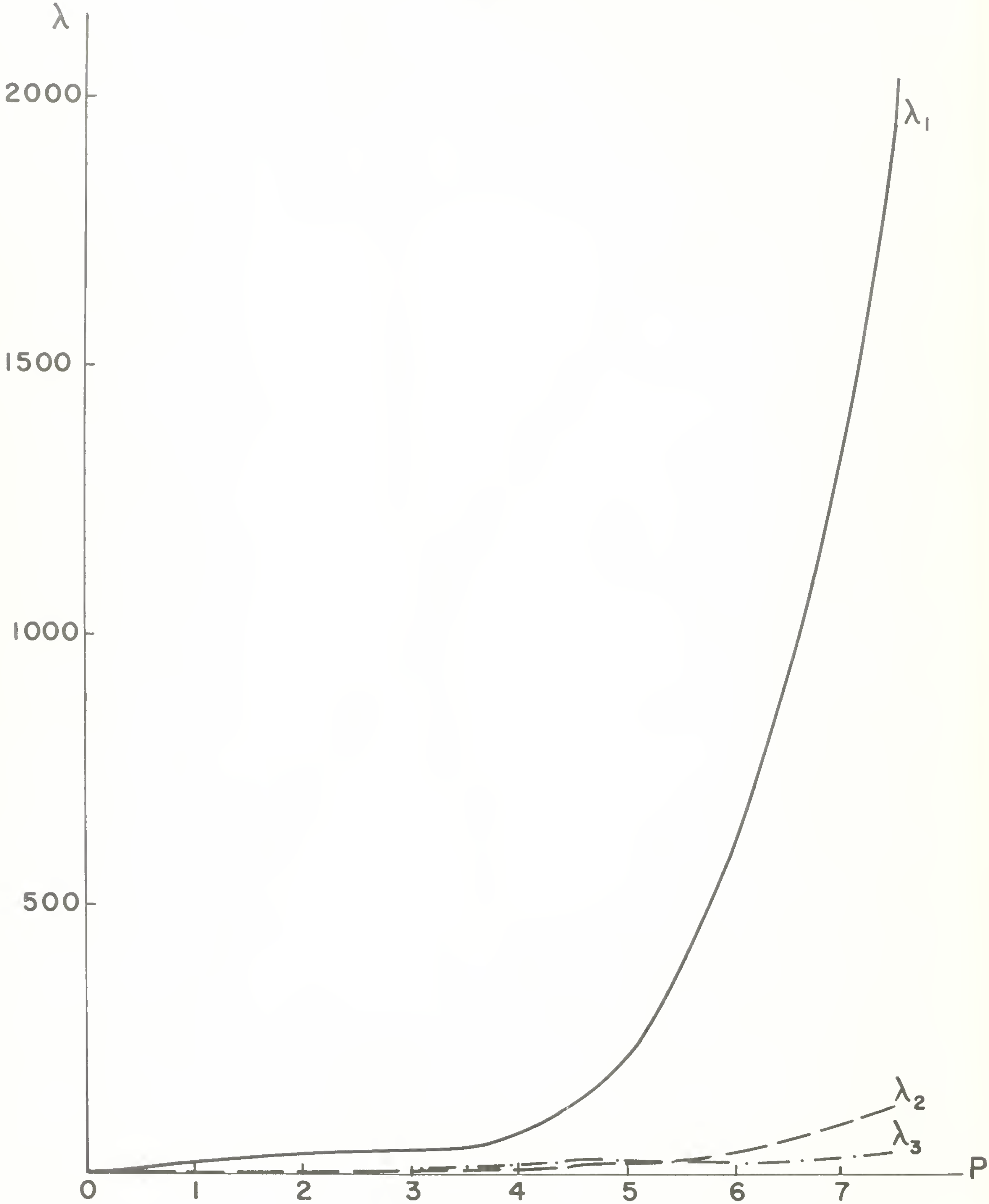

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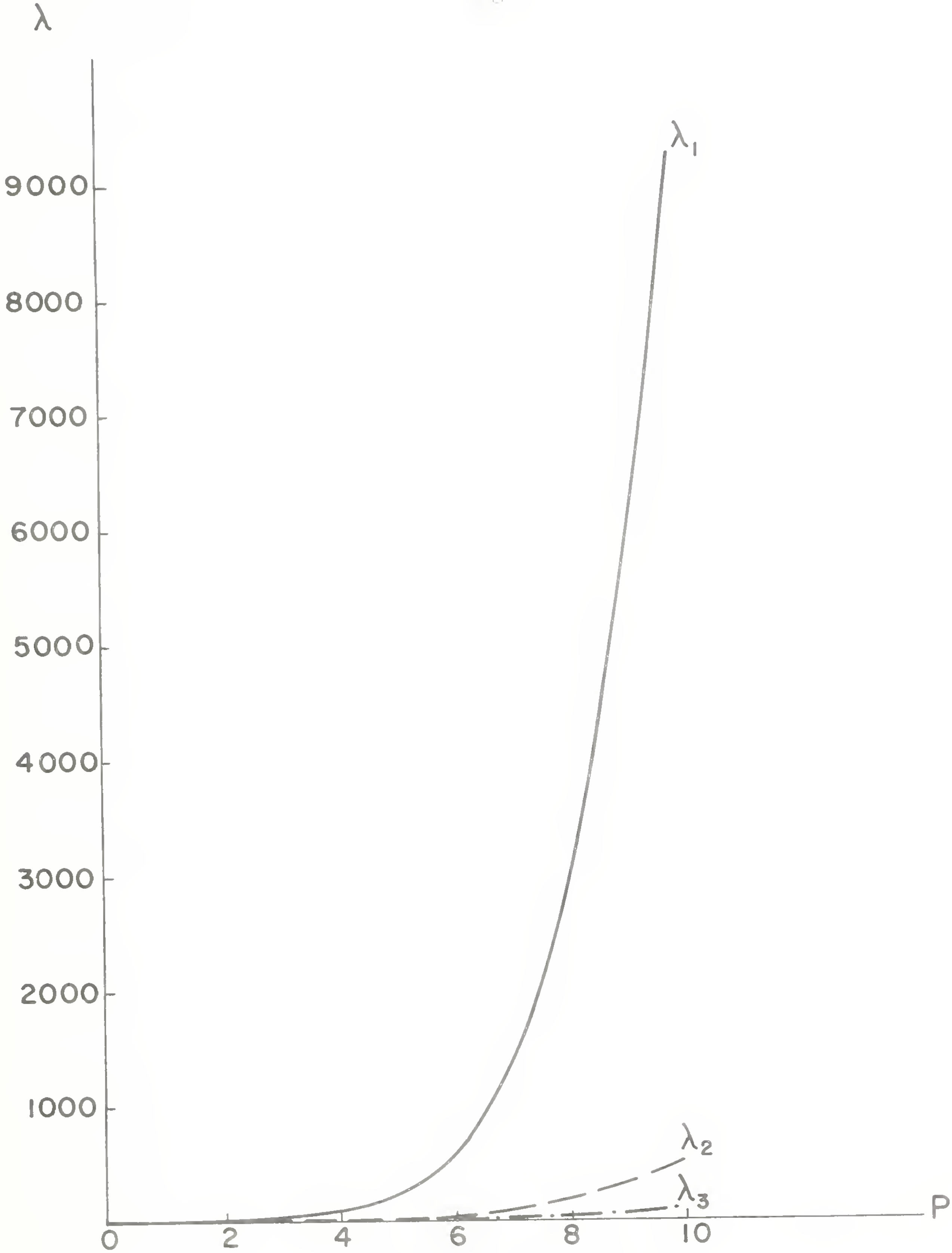

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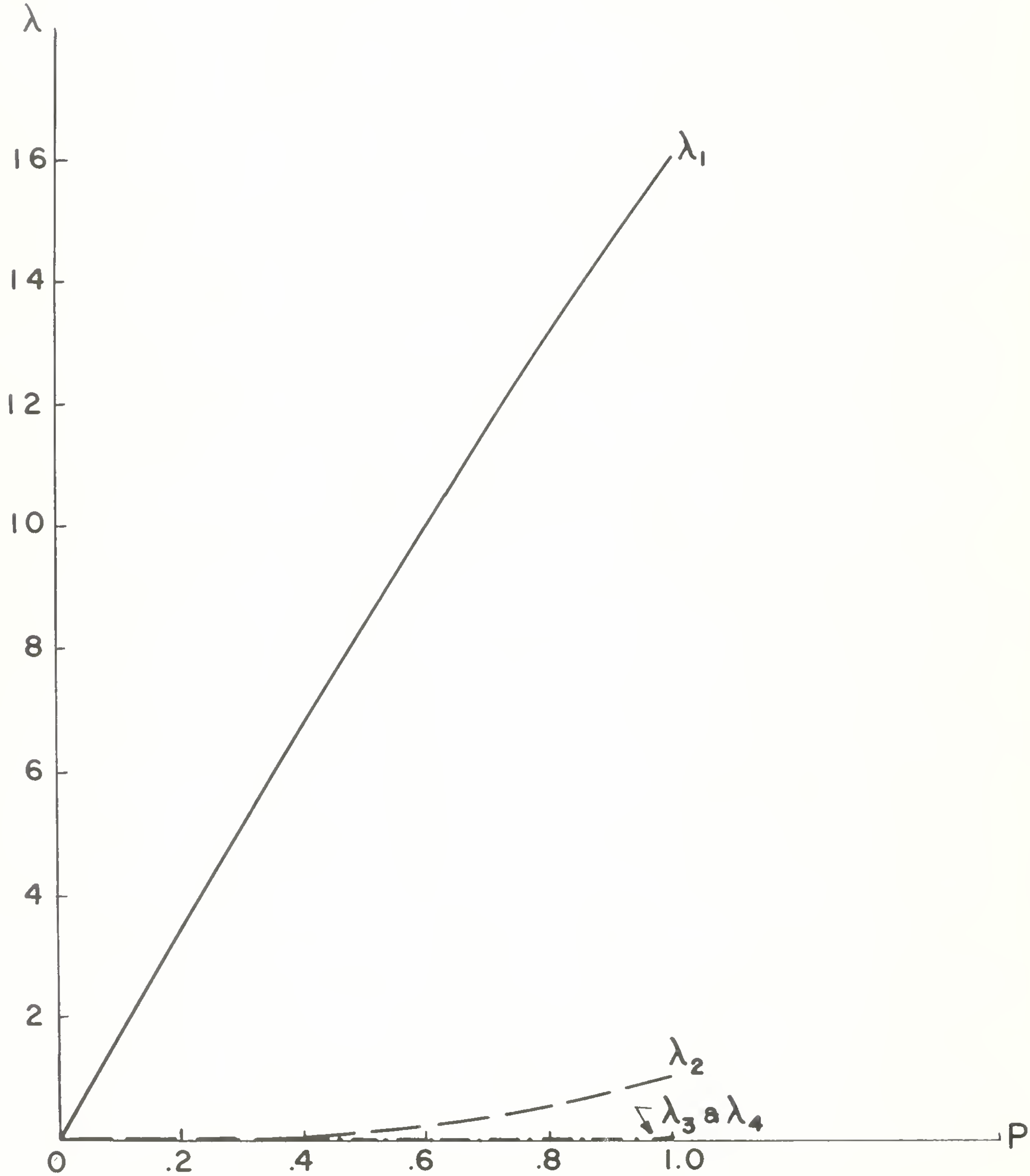
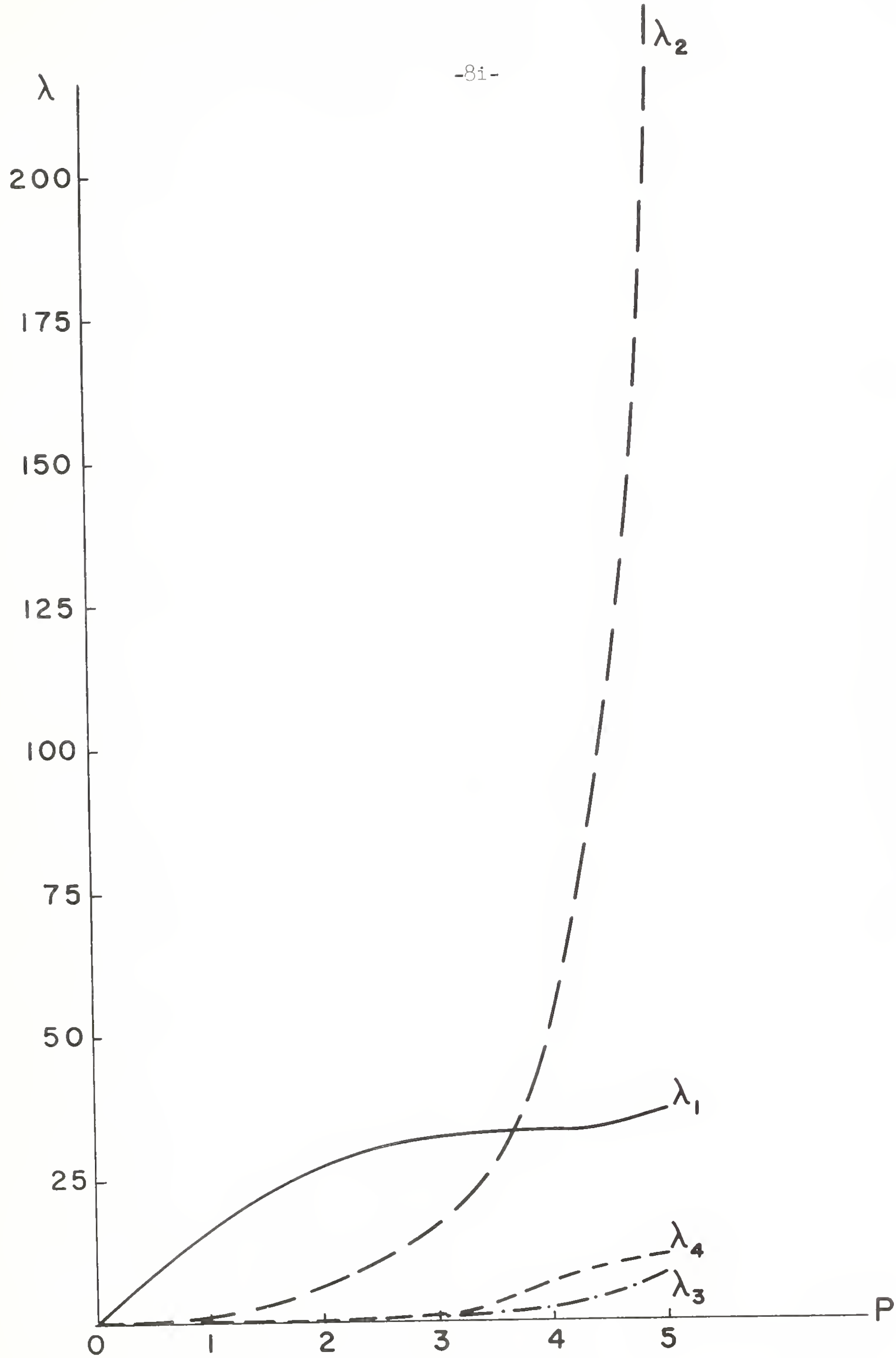
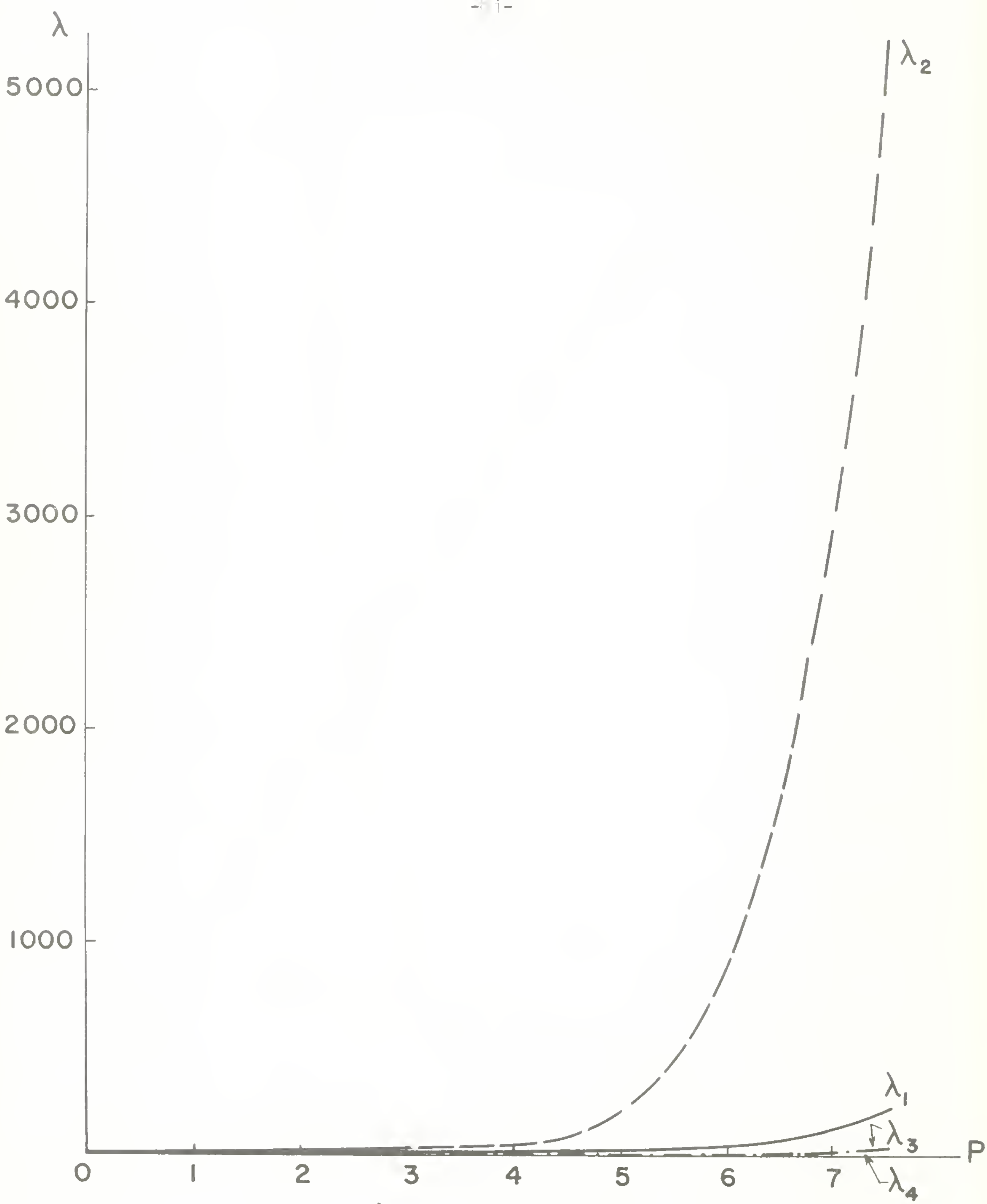


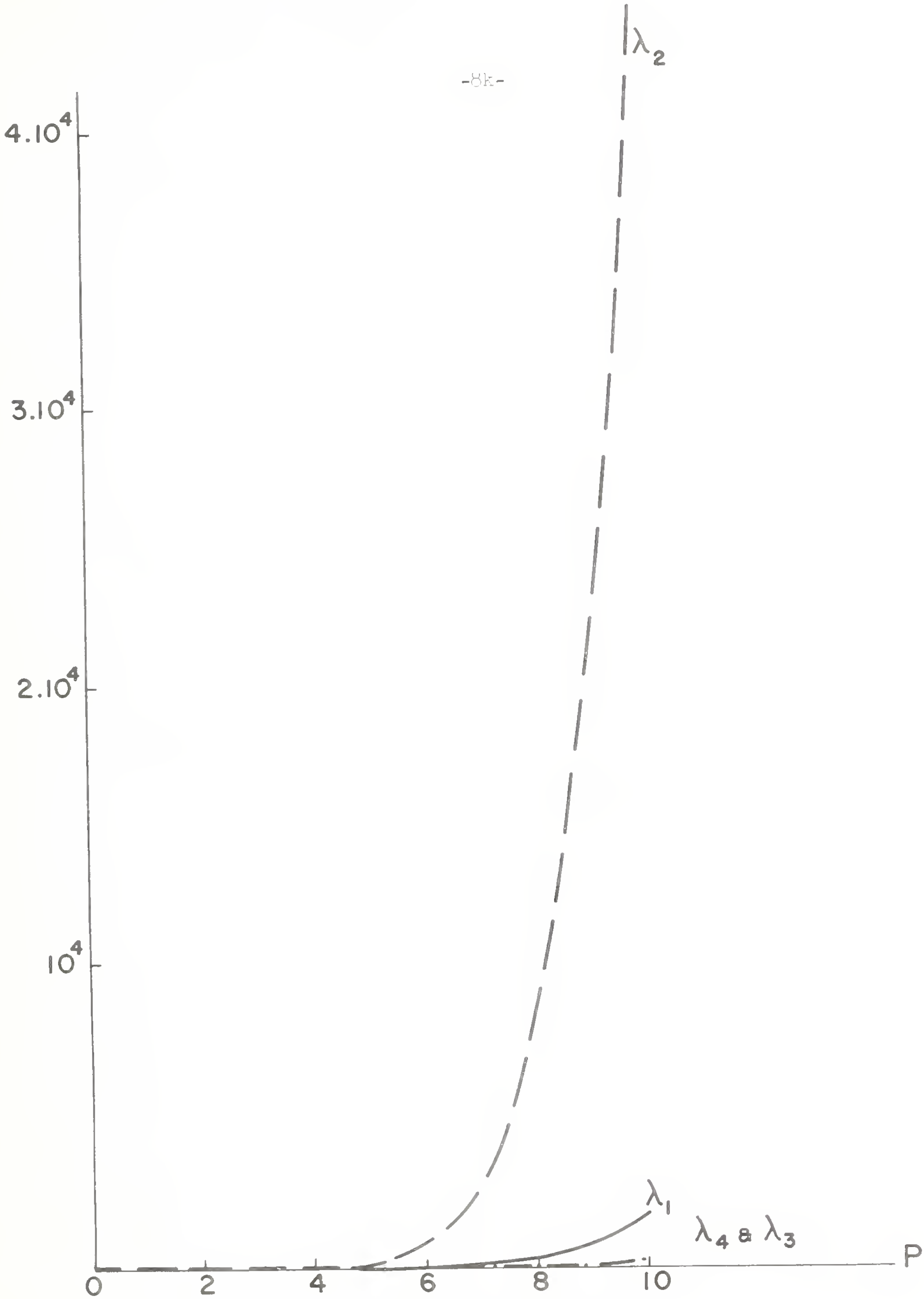
λ: Fourth Approximation



λ: Fourth Approximation



λ: Fourth Approximation



λ: Fourth Approximation

B. Equation 2.

A similar procedure is followed with equation (2).

$$\lambda^* f(y) = \int_0^1 \frac{\gamma}{2} \ell \frac{J_{1/2}[k\ell(x - y)]}{[k\ell(x - y)]^{1/2}} \sqrt{1 - x^2} \sqrt{1 - y^2} f(x)\, dx$$

First we use the identity

$$\frac{J_{1/2}[P(x - y)]}{[P(x - y)]^{1/2}} = \sqrt{\frac{2}{\pi}} \frac{\sin[P(x - y)]}{P(x - y)}$$

and then expand in a double Taylor series:

$$\frac{\sin[P(x - y)]}{P(x - y)} = 1 - \frac{1}{3!}[P(x - y)]^2 + \frac{1}{5!}[P(x - y)]^4$$

$f(x)$ and $f(y)$ are expanded into:

$$f(x) = \sqrt{1 - x^2}\, \Sigma f_n x^n, \quad f(y) = \sqrt{1 - y^2}\, \Sigma f_n x^n$$

The approximation procedure used on Equation (1) is then repeated. The results obtained are:

First Approximation

$$\lambda^* f_0 = 4 \pi P$$

Second Approximation

(Letting $s = 4 \pi P$)

$$\lambda^* f_0 = s \left\{ \left[\frac{2}{3} - \frac{P^2}{45}\right] f_0 + \left[\frac{1}{4} - \frac{P^2}{72}\right] f_1 + \left[\frac{2}{15} - \frac{P^2}{105}\right] f_2 \right\}$$

$$\lambda^* f_1 = s \left\{ \frac{P^2}{12} f_0 + \frac{2P^2}{45} f_1 + \frac{P^2}{36} f_2 \right\}$$

$$\lambda^* f_2 = s \left\{ -\frac{P^2}{9} f_0 - \frac{P^2}{24} f_1 - \frac{P^2}{45} f_2 \right\}$$

Third Approximation

$$\lambda^* f_0 = s\left\{ \left[\frac{2}{3} - \frac{P^2}{45} + \frac{P^4}{2100}\right] f_0 + \left[\frac{1}{4} - \frac{P^2}{72} + \frac{P^4}{2880}\right] f_1 + \left[\frac{2}{15} - \frac{P^2}{105} + \frac{P^4}{3780}\right] f_2 + \left[\frac{1}{12} - \frac{P^2}{144} + \frac{P^4}{4800}\right] f_3 + \left[\frac{2}{35} - \frac{P^2}{189} + \frac{P^4}{1188}\right] f_4 \right\}$$

$$\lambda^* f_1 = s\left\{ \left[\frac{P^2}{12} - \frac{P^4}{360}\right] f_0 + \left[\frac{2P^2}{45} - \frac{P^4}{525}\right] f_1 + \left[\frac{P^2}{36} - \frac{P^4}{720}\right] f_2 + \left[\frac{2P^2}{105} - \frac{P^4}{945}\right] f_3 + \left[\frac{P^2}{72} - \frac{P^4}{1200}\right] f_4 \right\}$$

$$\lambda^* f_2 = s\left\{ \left[-\frac{P^2}{9} + \frac{P^4}{30}\right] f_0 + \left[-\frac{P^2}{24} + \frac{P^4}{48}\right] f_1 + \left[-\frac{P^2}{45} + \frac{P^4}{70}\right] f_2 + \left[-\frac{P^2}{72} + \frac{P^4}{96}\right] f_3 + \left[-\frac{P^2}{105} + \frac{P^4}{126}\right] f_4 \right\}$$

$$\lambda^* f_3 = s\left\{ -\frac{P^4}{120} f_0 - \frac{P^4}{225} f_1 - \frac{P^4}{360} f_2 - \frac{P^4}{525} f_3 - \frac{P^4}{720} f_4 \right\}$$

$$\lambda^* f_4 = s\left\{ \frac{P^4}{180} f_0 + \frac{P^4}{480} f_1 + \frac{P^4}{900} f_2 + \frac{P^4}{1440} f_3 + \frac{P^4}{2100} f_4 \right\}$$

Fourth Approximation

$$
\begin{aligned}
\lambda^* f_0 = s\Big\{ & \left[\frac{2}{3} - \frac{P^2}{45} + \frac{P^4}{2100} - \frac{P^6}{158760}\right] f_0 \\
& + \left[\frac{1}{4} - \frac{P^2}{72} + \frac{P^4}{2880} - \frac{P^6}{201600}\right] f_1 \\
& + \left[\frac{2}{15} - \frac{P^2}{105} + \frac{P^4}{3780} - \frac{P^6}{249480}\right] f_2 \\
& + \left[\frac{1}{12} - \frac{P^2}{144} + \frac{P^4}{4800} - \frac{P^6}{302400}\right] f_3 \\
& + \left[\frac{2}{35} - \frac{P^2}{189} + \frac{P^4}{5940} - \frac{P^6}{360360}\right] f_4 \\
& + \left[\frac{1}{24} - \frac{P^2}{240} + \frac{P^4}{7200} - \frac{P^6}{423360}\right] f_5 \\
& + \left[\frac{2}{63} - \frac{P^2}{297} + \frac{P^4}{8580} - \frac{P^6}{491400}\right] f_6 \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\lambda^* f_1 = s\Big\{ & \left[\frac{P^2}{12} - \frac{P^4}{360} + \frac{P^6}{20160}\right] f_0 + \left[\frac{2P^2}{45} - \frac{P^4}{525} + \frac{P^6}{26460}\right] f_1 \\
& + \left[\frac{P^2}{36} - \frac{P^4}{720} + \frac{P^6}{33600}\right] f_2 + \left[\frac{2P^2}{105} - \frac{P^4}{945} + \frac{P^6}{41580}\right] f_3 \\
& + \left[\frac{P^2}{72} - \frac{P^4}{1200} + \frac{P^6}{50400}\right] f_4 + \left[\frac{2P^2}{189} - \frac{P^4}{1485} + \frac{P^6}{60060}\right] f_5 \\
& + \left[\frac{P^2}{120} - \frac{P^4}{1800} + \frac{P^6}{70560}\right] f_6 \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\lambda^* f_2 = s\Big\{ & \left[-\frac{P^2}{9} + \frac{P^4}{150} - \frac{P^6}{5880}\right] f_0 + \left[-\frac{P^2}{24} + \frac{P^4}{240} - \frac{P^6}{8064}\right] f_1 \\
& + \left[-\frac{P^2}{45} + \frac{P^4}{350} - \frac{P^6}{10584}\right] f_2 + \left[-\frac{P^2}{72} + \frac{P^4}{480} - \frac{P^6}{13440}\right] f_3 \\
& + \left[-\frac{P^2}{105} + \frac{P^4}{630} - \frac{P^6}{16632}\right] f_4 + \left[-\frac{P^2}{144} + \frac{P^4}{800} - \frac{P^6}{20160}\right] f_5 \\
& + \left[-\frac{P^2}{189} + \frac{P^4}{990} - \frac{P^6}{24024}\right] f_6 \Big\}
\end{aligned}
$$

$$\lambda^* f_3 = s\left\{\left[-\frac{P^4}{120}+\frac{P^6}{3024}\right]f_0 + \left[-\frac{P^4}{225}+\frac{P^6}{4410}\right]f_1 + \left[-\frac{P^4}{360}+\frac{P^6}{6024}\right]f_2 + \left[-\frac{P^4}{525}+\frac{P^6}{8938}\right]f_3 + \left[-\frac{P^4}{720}+\frac{P^6}{10080}\right]f_4 + \left[-\frac{P^4}{945}+\frac{P^6}{12474}\right]f_5 + \left[-\frac{P^4}{1200}+\frac{P^6}{15120}\right]f_6\right\}$$

$$\lambda^* f_4 = s\left\{\left[\frac{P^4}{180}-\frac{P^6}{2520}\right]f_0 + \left[\frac{P^4}{480}-\frac{P^6}{4032}\right]f_1 + \left[\frac{P^4}{900}-\frac{P^6}{5880}\right]f_2 + \left[\frac{P^4}{1440}-\frac{P^6}{8064}\right]f_3 + \left[\frac{P^4}{2100}-\frac{P^6}{10584}\right]f_4 + \left[\frac{P^4}{2880}-\frac{P^6}{13440}\right]f_5 + \left[\frac{P^4}{3780}-\frac{P^6}{16632}\right]f_6\right\}$$

$$\lambda^* f_5 = s\left\{\frac{P^6}{3360}f_0 + \frac{P^6}{6300}f_1 + \frac{P^6}{10080}f_2 + \frac{P^6}{14700}f_3 + \frac{P^6}{20160}f_4 + \frac{P^6}{26460}f_5 + \frac{P^6}{33600}f_6\right\}$$

$$\lambda^* f_6 = s\left\{-\frac{P^6}{7560}f_0 - \frac{P^6}{20160}f_1 - \frac{P^6}{37800}f_2 - \frac{P^6}{60480}f_3 - \frac{P^6}{88200}f_4 - \frac{P^6}{120160}f_5 - \frac{P^6}{158760}f_6\right\}$$

The numerical results and graphs now follow.

Equation 2

First Approximation $\lambda^* = 8.3780\ P$

P	λ^*	P	λ^*
0.1	0.8370	0.6	5.0220
0.2	1.6740	0.7	5.8590
0.3	2.5110	0.8	6.6960
0.4	3.3480	0.9	7.5330
0.5	4.1850	1.0	8.370

First Approximation $\lambda^* = 8.3780$ P Continued

P	λ^*	P	λ^*
1.1	9.216	4.6	38.539
1.2	10.054	4.7	39.377
1.3	10.891	4.8	40.214
1.4	11.729	4.9	41.522
1.5	12.567	5.0	41.850
1.6	13.405	5.1	42.728
1.7	14.243	5.2	43.566
1.8	15.080	5.3	44.403
1.9	15.918	5.4	45.241
2.0	16.740	5.5	46.079
2.1	17.594	5.6	46.917
2.2	18.432	5.7	47.755
2.3	19.269	5.8	48.592
2.4	20.107	5.9	49.430
2.5	20.945	6.0	50.220
2.6	21.783	6.1	51.106
2.7	22.621	6.2	51.944
2.8	23.458	6.3	52.781
2.9	24.296	6.4	53.619
3.0	25.110	6.5	54.457
3.1	25.972	6.6	55.295
3.2	26.810	6.7	56.133
3.3	27.647	6.8	56.970
3.4	28.485	6.9	57.808
3.5	29.323	7.0	58.590
3.6	30.161	7.1	59.484
3.7	30.999	7.2	60.322
3.8	31.836	7.3	61.159
3.9	32.674	7.4	61.997
4.0	33.480	7.5	62.835
4.1	34.350	7.6	63.673
4.2	35.188	7.7	64.511
4.3	36.025	7.8	65.348
4.4	36.836	7.9	66.186
4.5	37.701	8.0	66.960

First Approximation $\lambda^{*} = 8.3780\ P$ Continued

P	λ^{*}	P	λ^{*}
8.1	67.862	9.1	76.240
8.2	68.700	9.2	77.078
8.3	69.537	9.3	77.915
8.4	70.375	9.4	78.753
8.5	71.213	9.5	79.591
8.6	72.051	9.6	80.429
8.7	72.889	9.7	81.267
8.8	73.726	9.8	82.104
8.9	74.564	9.9	82.942
9.0	75.330	10.0	83.700

Equation 2

Second Approximation

	λ_1^{*}	λ_2^{*}
P = 0.1	0.8376	0.0002
0.2	1.6742	0.0013
0.3	2.5088	0.0045
0.4	3.3404	0.0106
0.5	4.1681	0.0207
0.6	4.9908	0.0358
0.7	5.8076	0.0569
0.8	6.6175	0.0849
0.9	7.4195	0.1208
1.0	8.2128	0.1657
1.1	8.9963	0.2205
1.2	9.7691	0.2862
1.3	10.5303	0.3638
1.4	11.2790	0.4543
1.5	12.0144	0.5586
1.6	12.7357	0.6778
1.7	13.4418	0.8128
1.8	14.1322	0.9645

Second Approximation Continued

	λ_1^*	λ_2^*
P = 1.9	14.8059	1.1340
2.0	15.4623	1.3221
2.1	16.1007	1.5299
2.2	16.7204	1.7583
2.3	17.3207	2.0081
2.4	17.9013	2.2805
2.5	18.4614	2.5761
2.6	19.0008	2.8961
2.7	19.5190	3.2411
2.8	20.0158	3.6122
2.9	20.4909	4.0101
3.0	20.9444	4.4356
3.1	21.3762	4.8895
3.2	21.7864	5.3726
3.3	22.1755	5.8855
3.4	22.5438	6.4288
3.5	22.8920	7.0031
3.6	23.2211	7.6088
3.7	23.5321	8.2463
3.8	23.8266	8.9156
3.9	24.1063	9.6167
4.0	24.3735	10.3493
4.1	24.6308	11.1127
4.2	24.8817	11.9000
4.3	25.1303	12.7267
4.4	25.3818	13.5728
4.5	25.6426	14.4405
4.6	25.9208	15.3245
4.7	26.2266	16.2177
4.8	26.5724	17.1106
4.9	26.9740	17.9909
5.0	27.4494	18.8438
5.1	28.0185	19.6526
5.2	28.7006	20.4015

Second Approximation Continued

	λ_1^*	λ_2^*
P = 5.3	29.5109	21.0789
5.4	30.4573	21.6797
5.5	31.5402	22.2073
5.6	32.7532	22.6797
5.7	34.0870	23.0826
5.8	35.5314	23.4559
5.9	37.0771	23.8024
6.0	38.7165	24.1324
6.1	40.4437	24.4539
6.2	42.2544	24.7735
6.3	44.1452	25.0963
6.4	46.1139	25.4265
6.5	48.1590	25.7671
6.6	50.2794	26.1207
6.7	52.4747	26.4892
6.8	54.7444	26.8742
6.9	57.0887	27.2767
7.0	59.5078	27.6977
7.1	62.0019	28.1379
7.2	64.5716	28.5979
7.3	67.2173	29.0780
7.4	69.9398	29.5786
7.5	72.7397	30.0999
7.6	75.6178	30.6420
7.7	78.5748	31.2052
7.8	81.6115	31.7894
7.9	84.7289	32.3947
8.0	87.9278	33.0213
8.1	91.2090	33.6691
8.2	94.5735	34.3381
8.3	98.0221	35.0284
8.4	101.5559	35.7399
8.5	105.1757	36.4728
8.6	108.8825	37.2270

Second Approximation Continued

	λ_1^*	λ_2^*
P = 8.7	112.6773	38.0026
8.8	116.5610	38.7996
8.9	120.5345	39.6180
9.0	124.5989	40.4579
9.1	128.7551	41.3194
9.2	133.0042	42.2025
9.3	137.3470	43.1072
9.4	141.7846	44.0337
9.5	126.3179	44.9820
9.6	150.9480	45.9522
9.7	155.6759	46.9444
9.8	160.5025	47.9587
9.9	165.4288	48.9952
10.0	170.4559	50.0540

Third Approximation

P	λ_1^*	λ_2^*	λ_3^*
0.1	0.8376	0.0002	0.0000
0.2	1.6742	0.0013	0.0000
0.3	2.5088	0.0045	0.0000
0.4	3.3405	0.0106	0.0000
0.5	4.1682	0.0206	0.0000
0.6	4.9910	0.0355	0.0001
0.7	5.8080	0.0562	0.0001
0.8	6.6183	0.0835	0.0002
0.9	7.4210	0.1184	0.0004
1.0	8.2153	0.1616	0.0007
1.1	9.0002	0.2140	0.0011
1.2	9.7752	0.2762	0.0018
1.3	10.5394	0.3488	0.0027
1.4	11.2922	0.4326	0.0039
1.5	12.0328	0.5281	0.0056
1.6	12.7609	0.6356	0.0078

Third Approximation Continued

P	λ_1^*	λ_2^*	λ_3^*
1.7	13.4757	0.7558	0.0107
1.8	14.1769	0.8887	0.0145
1.9	14.8640	1.0348	0.0193
2.0	15.5366	1.1942	0.0253
2.1	16.1945	1.3669	0.0328
2.2	16.8375	1.5530	0.0420
2.3	17.4653	1.7523	0.0533
2.4	18.0779	1.9646	0.0671
2.5	18.6751	2.1896	0.0837
2.6	19.2571	2.4270	0.1037
2.7	19.8239	2.6761	0.1275
2.8	20.3756	2.9365	0.1557
2.9	20.9124	3.2072	0.1891
3.0	21.4346	3.4876	0.2282
3.1	21.9425	3.7767	0.2740
3.2	22.4364	4.0735	0.3274
3.3	22.9167	4.3767	0.3892
3.4	23.3840	4.6853	0.4607
3.5	23.8387	4.9978	0.5429
3.6	24.2814	5.3129	0.6371
3.7	24.7127	5.6291	0.7448
3.8	25.1332	5.9450	0.8674
3.9	25.5438	6.2588	1.0064
4.0	25.9451	6.5691	1.1637
4.1	26.3380	6.8743	1.3410
4.2	26.7234	7.1728	1.5401
4.3	27.1022	7.4633	1.7630
4.4	27.4757	7.7442	2.0117
4.5	27.8449	8.0146	2.2882
4.6	28.2113	8.2734	2.5944
4.7	28.5763	8.5202	2.9323
4.8	28.9419	8.7548	3.3035
4.9	29.3099	8.9779	3.7093
5.0	29.6829	9.1909	4.1505

Third Approximation Continued

P	λ_1^*	λ_2^*	λ_3^*
5.1	30.0636	9.3963	4.6270
5.2	30.4554	9.5985	5.1372
5.3	30.8623	9.8041	5.6775
5.4	31.2891	10.0234	6.2408
5.5	31.7416	10.2715	6.8147
5.6	32.2268	10.5706	7.3797
5.7	32.7530	10.9488	7.9097
5.8	33.3305	11.4333	8.3782
5.9	34.9713	12.0374	8.7713
6.0	34.6899	12.7529	9.0946
6.1	35.5034	13.5562	9.3666
6.2	36.4312	14.4195	9.6076
6.3	37.4954	15.3164	9.8340
6.4	38.7199	16.2232	10.0576
6.5	40.1294	17.1186	10.2867
6.6	41.7486	17.9841	10.5273
6.7	43.6005	18.8044	10.7832
6.8	45.7053	19.5685	11.0572
6.9	48.0800	20.2691	11.3515
7.0	50.7386	20.9035	11.6678
7.1	53.6919	21.4718	12.0068
7.2	56.9491	21.9772	12.3696
7.3	60.5181	22.4243	12.7565
7.4	64.4068	22.8185	13.1681
7.5	68.6230	23.1659	13.6048
7.6	73.1754	23.4724	14.0665
7.7	78.0730	23.7433	14.5532
7.8	83.3263	23.9844	15.0646
7.9	88.9467	24.2014	15.6005
8.0	94.9459	24.3989	16.1597
8.1	101.3373	24.5830	16.7414
8.2	108.1346	24.7593	17.3437
8.3	115.3524	24.9345	17.9632

Third Approximation Continued

P	λ_1^*	λ_2^*	λ_3^*
8.4	123.0063	25.1164	18.5964
8.5	131.1121	25.3144	19.2365
8.6	139.6868	25.5407	19.8749
8.7	148.7480	25.8110	20.5008
8.8	158.3133	26.1428	21.0966
8.9	168.4018	26.5574	21.6480
9.0	179.0328	27.0719	22.1393
9.1	190.2259	27.6980	22.5626
9.2	202.0019	28.4374	22.9201
9.3	214.3814	29.2837	23.2198
9.4	227.3860	30.2270	23.4752
9.5	241.0380	31.2570	23.7006
9.6	255.3600	32.3654	23.9073
9.7	270.3758	33.5458	24.1061
9.8	286.1080	34.7939	24.2994
9.9	302.5821	36.1067	24.4992
10.0	319.8223	37.4828	24.7064

Equation 2

Fourth Approximation

P	λ_1^*	λ_2^*	λ_3^*	λ_4^*
0.1	0.8376	0.0002	0.0000	0.0000
0.2	1.6742	0.0013	0.0000	
0.3	2.5088	0.0045	0.0000	
0.4	3.3405	0.0106	0.0000	
0.5	4.1682	0.0206	0.0000	
0.6	4.9910	0.0355	0.0001	
0.7	5.8080	0.0562	0.0001	
0.8	6.6183	0.0835	0.0002	
0.9	7.4210	0.1184	0.0004	
1.0	8.2153	0.1616	0.0007	
1.1	9.0002	0.2140	0.0011	
1.2	9.7752	0.2762	0.0018	
1.3	10.5394	0.3489	0.0027	
1.4	11.2921	0.4327	0.0039	
1.5	12.0328	0.5282	0.0055	0.0000

P	λ_1^*	λ_2^*	λ_3^*	λ_4^*
1.6	12.7608	0.6358	0.0077	0.0000
1.7	13.4756	0.7560	0.0106	
1.8	14.1767	0.8890	0.0143	
1.9	14.8637	1.0353	0.0190	
1.0	15.5363	1.1948	0.0248	
2.1	16.1941	1.3678	0.0321	
2.2	16.8369	1.5542	0.0411	
2.3	17.4645	1.7540	0.0521	
2.4	18.0769	1.9669	0.0654	
2.5	18.6738	2.1926	0.0815	
2.6	19.2554	2.4309	0.1007	
2.7	19.8217	2.6812	0.1236	
2.8	20.3728	2.9430	0.1506	
2.9	20.9088	3.2155	0.1825	
3.0	21.4301	3.4980	0.2198	
3.1	21.9369	3.7897	0.2634	
3.2	22.4296	4.0895	0.3140	
3.3	22.9085	4.3964	0.3725	
3.4	23.3740	4.7093	0.4400	
3.5	23.8266	5.0269	0.5174	
3.6	24.2669	5.3479	0.6059	
3.7	24.6954	5.6709	0.7068	
3.8	25.1128	5.9945	0.8213	
3.9	25.5196	6.3173	0.9510	
4.0	25.9166	6.6376	1.0974	
4.1	26.3045	6.9541	1.2619	
4.2	26.6841	7.2652	1.4464	
4.3	27.0563	7.5694	1.6525	
4.4	27.4221	7.8654	1.8821	
4.5	27.7823	8.1520	2.1369	
4.6	28.1383	8.4280	2.4188	
4.7	28.4912	8.6927	2.7295	
4.8	28.8425	8.9454	3.0707	
4.9	29.1938	9.1861	3.4440	
5.0	29.5470	9.4153	3.8503	0.0000

P	λ_1^*	λ_2^*	λ_3^*	λ_4^*
5.1	29.9043	9.6342	4.2905	0.0000
5.2	30.2682	9.8449	4.7643	
5.3	30.6416	10.0510	5.2707	
5.4	31.0283	10.2579	5.8066	
5.5	31.4324	10.4739	6.3665	
5.6	31.8590	10.7108	6.9406	
5.7	32.3141	10.9860	7.5140	
5.8	32.8051	11.3224	8.0654	
5.9	33.3404	11.7455	9.5700	
6.0	33.9309	12.2746	9.0081	
6.1	34.5883	12.9125	9.3748	
6.2	35.3272	13.6442	9.6812	
6.3	36.1640	14.4456	9.9454	
6.4	37.1174	15.2907	10.1850	
6.5	38.2075	16.1588	10.4137	
6.6	39.4558	17.0195	10.6417	0.0000
6.7	40.8835	17.8632	10.8763	0.0001
6.8	42.5111	18.6711	11.1224	0.0001
6.9	44.3567	19.4313	11.3834	0.0001
7.0	46.4359	20.1351	11.6625	0.0001
7.1	48.7609	20.7779	11.9611	0.0001
7.2	51.3415	21.3585	12.2807	0.0001
7.3	54.1848	21.8784	12.6223	0.0001
7.4	57.2969	22.3409	12.9866	0.0002
7.5	60.6825	22.7506	13.3743	0.0002
7.6	64.3466	23.1129	13.7855	0.0002
7.7	68.2939	23.4331	14.2206	0.0002
7.8	72.5297	23.7167	14.6793	0.0003
7.9	77.0603	23.9691	15.1619	0.0003
8.0	81.8917	24.1952	15.6675	0.0004
8.1	87.0314	24.4004	16.1957	0.0005
8.2	92.4871	24.5897	16.7457	0.0005
8.3	98.2671	24.7687	17.3157	0.0006
8.4	104.3803	24.9434	17.9036	0.0007
8.5	110.8360	25.1204	18.5057	0.0008

P	λ_1^*	λ_2^*	λ_3^*	λ_4^*
8.6	117.6438	25.3086	19.1177	0.0009
8.7	124.8139	25.5163	19.7322	0.0010
8.8	132.3565	25.7578	20.3396	0.0012
8.9	140.2827	26.0484	20.9283	0.0013
9.0	148.6032	26.4053	21.4830	0.0015
9.1	157.3294	26.8470	21.9894	0.0017
9.2	166.4726	27.3876	22.4363	0.0019
9.3	176.0444	28.0336	22.8192	0.0022
9.4	186.0568	28.7838	23.1441	0.0024
9.5	196.5216	29.6308	23.4189	0.0028
9.6	207.4510	30.5654	23.6595	0.0031
9.7	218.8571	31.5788	23.8753	0.0035
9.8	230.7524	32.5640	24.0758	0.0040
9.9	243.1494	33.8158	24.2713	0.0045
10.0	256.0599	35.0305	24.4651	0.0050

We have three checks on the accuracy of our results. First, the eigenvalues are continuous (monotonic) functions of the parameter. Second, the trace of each coefficient matrix equals the sum of its eigenvalues. Third, we find on examining our results that for a fixed value of P, the trace of each matrix remains invariant under successive approximation.

We now prove our last statement and give a necessary and sufficient condition for this to occur in more general circumstances. We denote the

kernel by $K(x, y)$. We integrate in a region $[-b, b]$ symmetric about the origin

$$\lambda\, f(y) = \int_{-b}^{b} K\,(x, y)\, f(x)\, dx$$

Taking a finite Taylor expansion, we obtain:

$$\lambda \sum_{i=1}^{N} f_i y^i = \int_{-b}^{b} \left(\sum_{m,i=1}^{N} K_{mi}\, x^m\, y^i \right) \left(\sum_{j=1}^{N} f_j x^j \right) dx$$

$$\text{therefore} \quad \lambda\, f_i = \int_{-b}^{b} \left(\sum_{m,j}^{N} K_{mi}\, f_j\, x^{m+j} \right) dx \quad i = 1, \ldots, N$$

$$= \sum_{m,j=1}^{N} b^{m+j+1} \left(\frac{1 + (-1)^{m+j}}{m + j + 1} \right) K_{mi}\, f_j$$

$$= \sum_{j=1}^{N} A_{ij}\, f_j$$

$$\text{where } A_{ij} = \sum_{m=1}^{N} b^{m+j+1} \left(\frac{1 + (-1)^{m+j}}{m + j + 1} \right) K_{mi}$$

$$(3) \quad \text{Trace} = \sum_{i=1}^{N} A_{ii} = \sum_{i,m=1}^{N} b^{m+i+1} \left(\frac{1 + (-1)^{m+i}}{m + i + 1} \right) K_{mi}$$

If (as is the case with our equation) we have a difference kernel, $K\,(x,y) = G\,(x-y)$

$$\text{therefore} \quad K_{mi} = \frac{(-1)^i \binom{m+i}{i} G^{(m+i)}\,(0)}{(m + i)\,!}$$

Thus we obtain:

$$\text{Trace} = \sum_{i,m=1}^{N} b^{m+i+1} \frac{\left[1 + (-1)^{m+i}\right] (-1)^i \binom{m+i}{i} G^{(m+i)}\,(0)}{(m + i + 1)\,!}$$

$$= \sum_{n=0}^{2N} \sum_{i+m=n} b^{n+1} \frac{\left[1 + (-1)^{n}\right] (-1)^i \binom{n}{i} G^{(n)}\,(0)}{(n + 1)\,!}$$

$$= \sum_{n=0}^{2N} b^{n+1} \frac{\left[1 + (-1)^{n}\right]}{(n+1)!} G^{(n)}(0) \sum_{i=0}^{n} (-1)^i \binom{n}{i}$$

$$= \sum_{n=0}^{2N} b^{n+1} \frac{\left[1 + (-1)^n\right] G^{(n)}(0)}{(n+1)!} \sum_{i=0}^{n} 1^{n-i} (-1)^i \binom{n}{i}$$

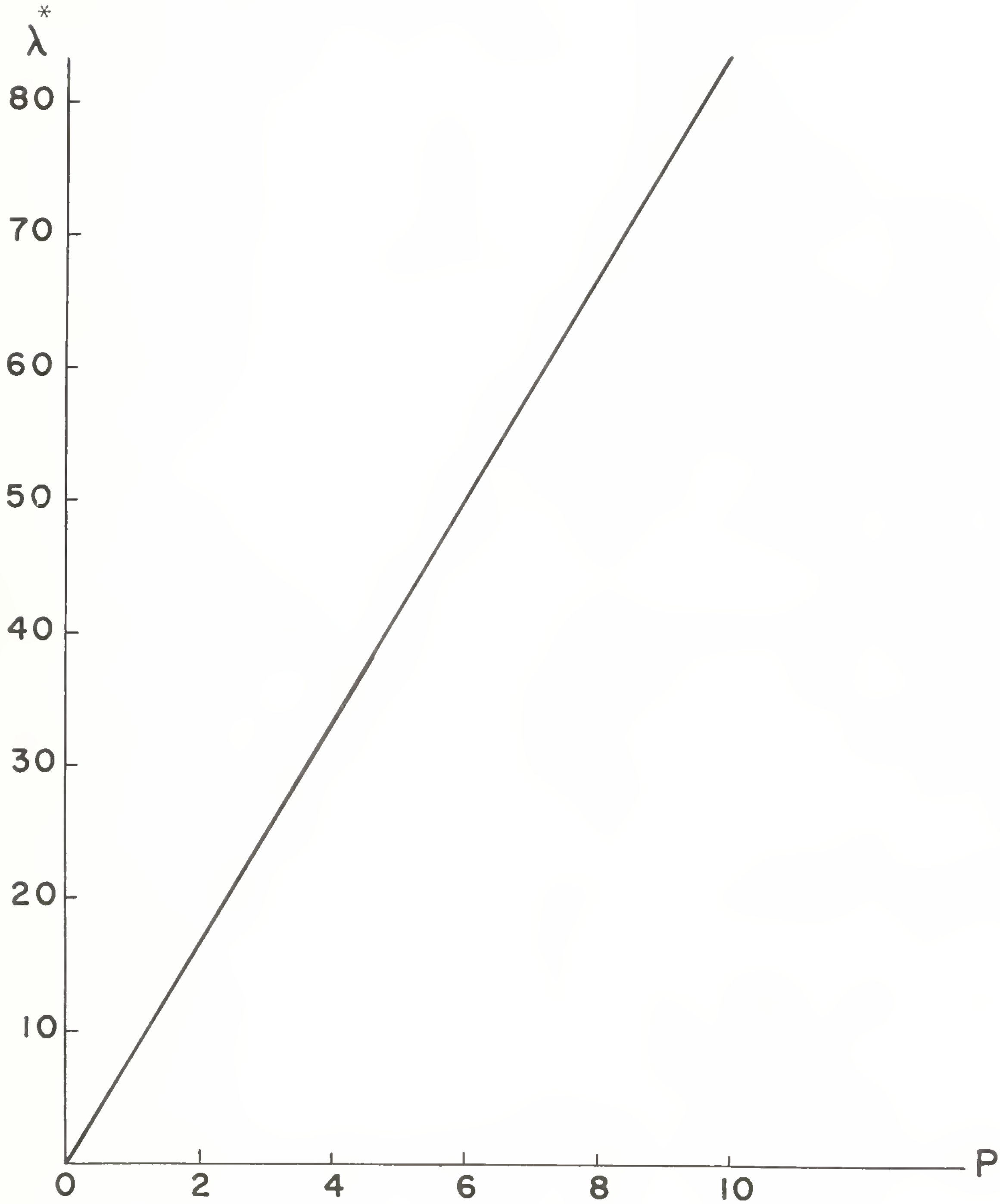
$$= \sum_{n=0}^{2N} b^{n+1} \frac{\left[1 + (-1)^n\right] G^{(n)}(0)}{(n+1)!} (1 - 1)^n$$

$$= 2\,b\ \ G(0)$$

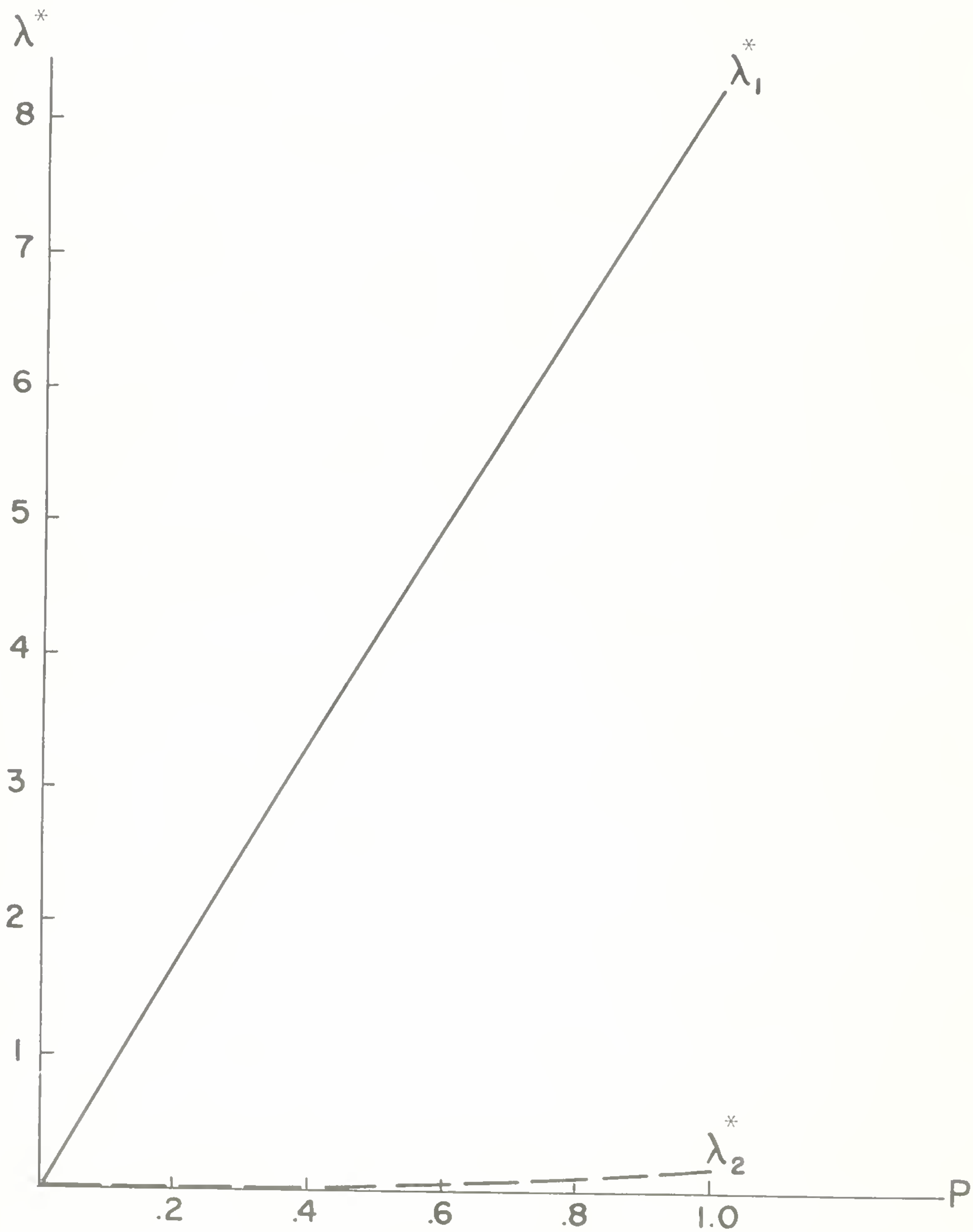
therefore Trace = constant

We note from (3) that the terms vanish for which $m + i$ = odd number. Thus a necessary and sufficient condition for the invariance of the trace under successive polynomial approximation is

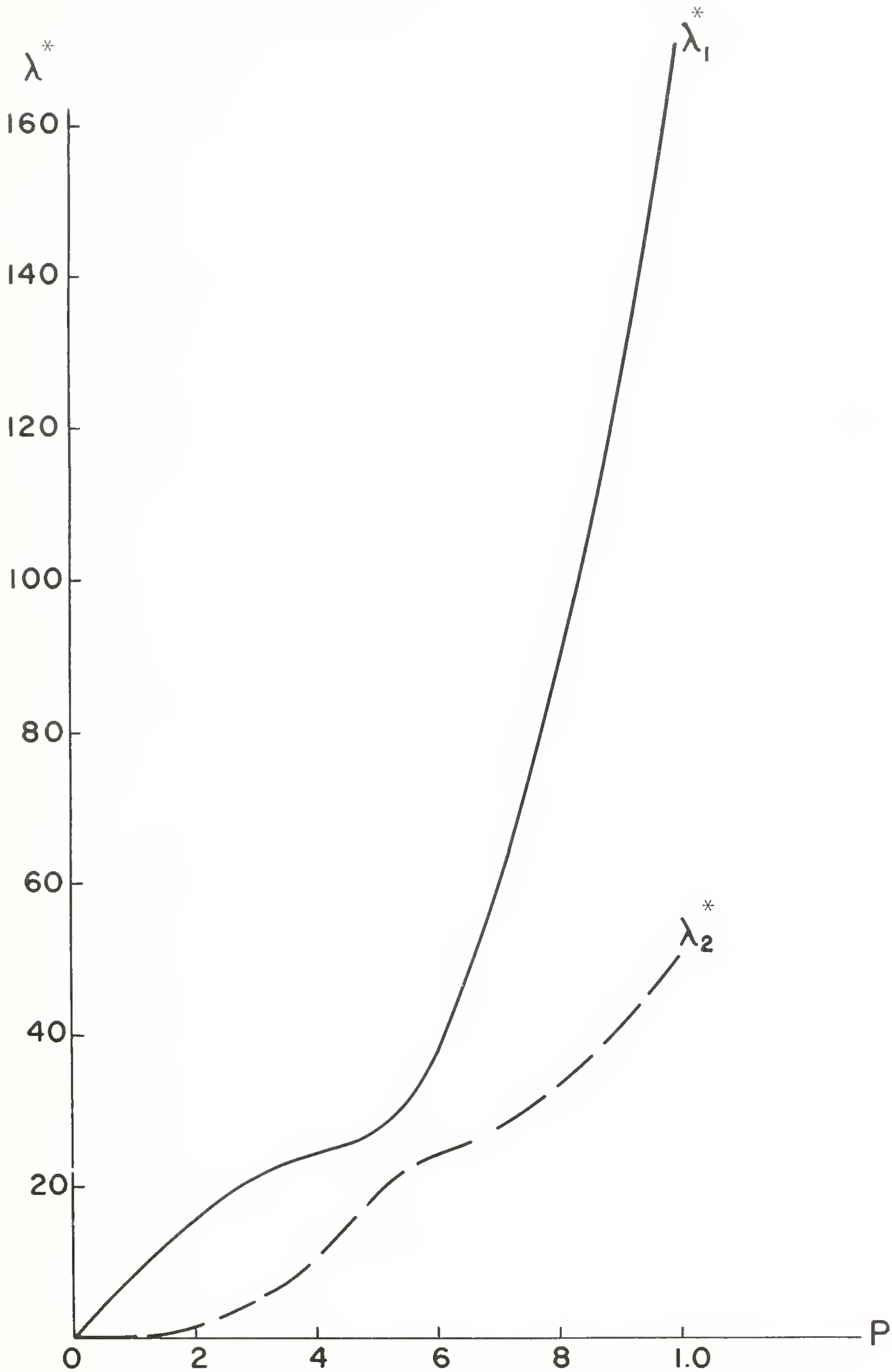
$$\sum_{m+i=n} \frac{1 + (-1)^{m+i}}{m + i} K_{mi} = 0 \text{ for } m + i \text{ even}, \quad m + 1 > 0.$$



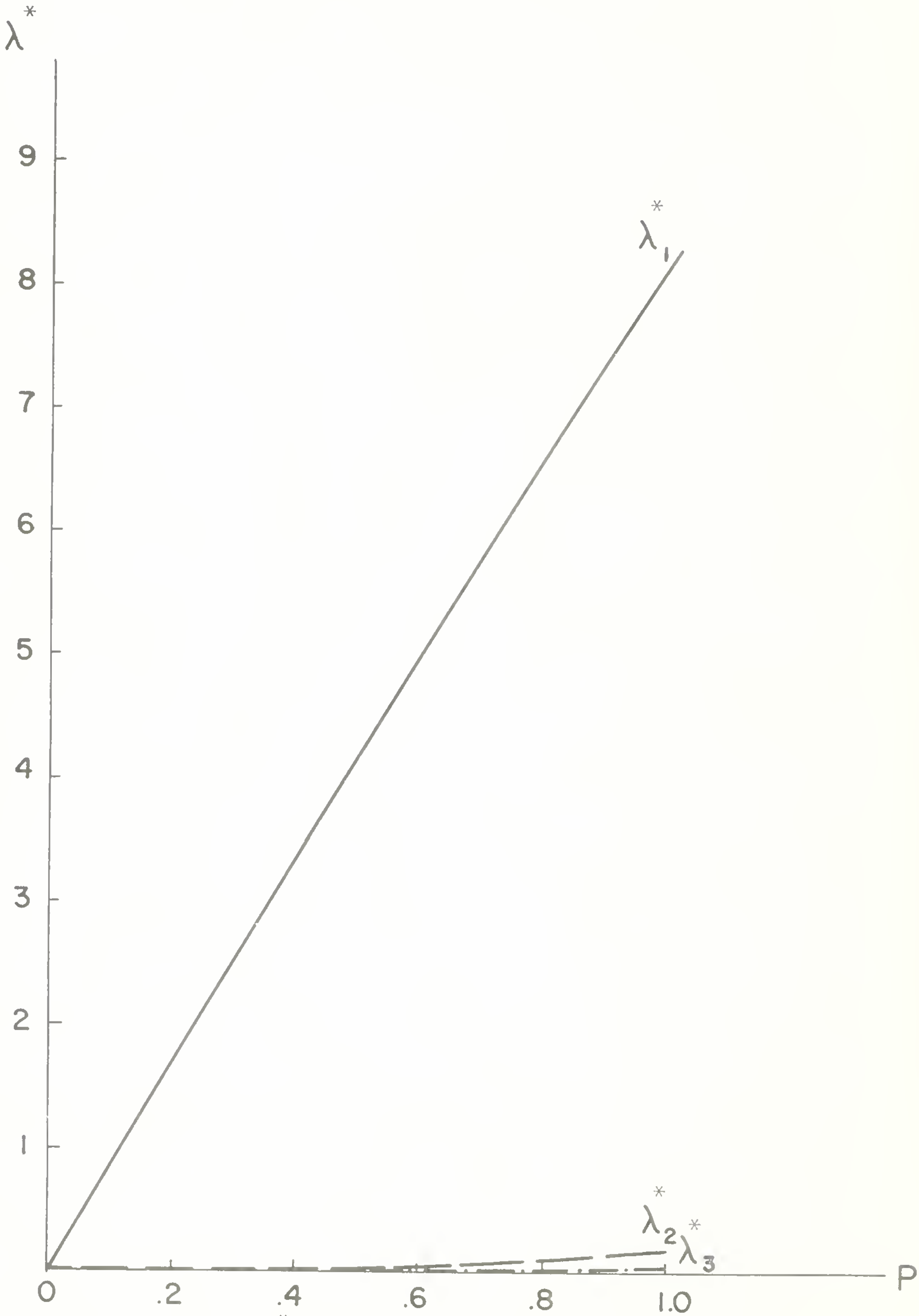
λ^{*}: First Approximation



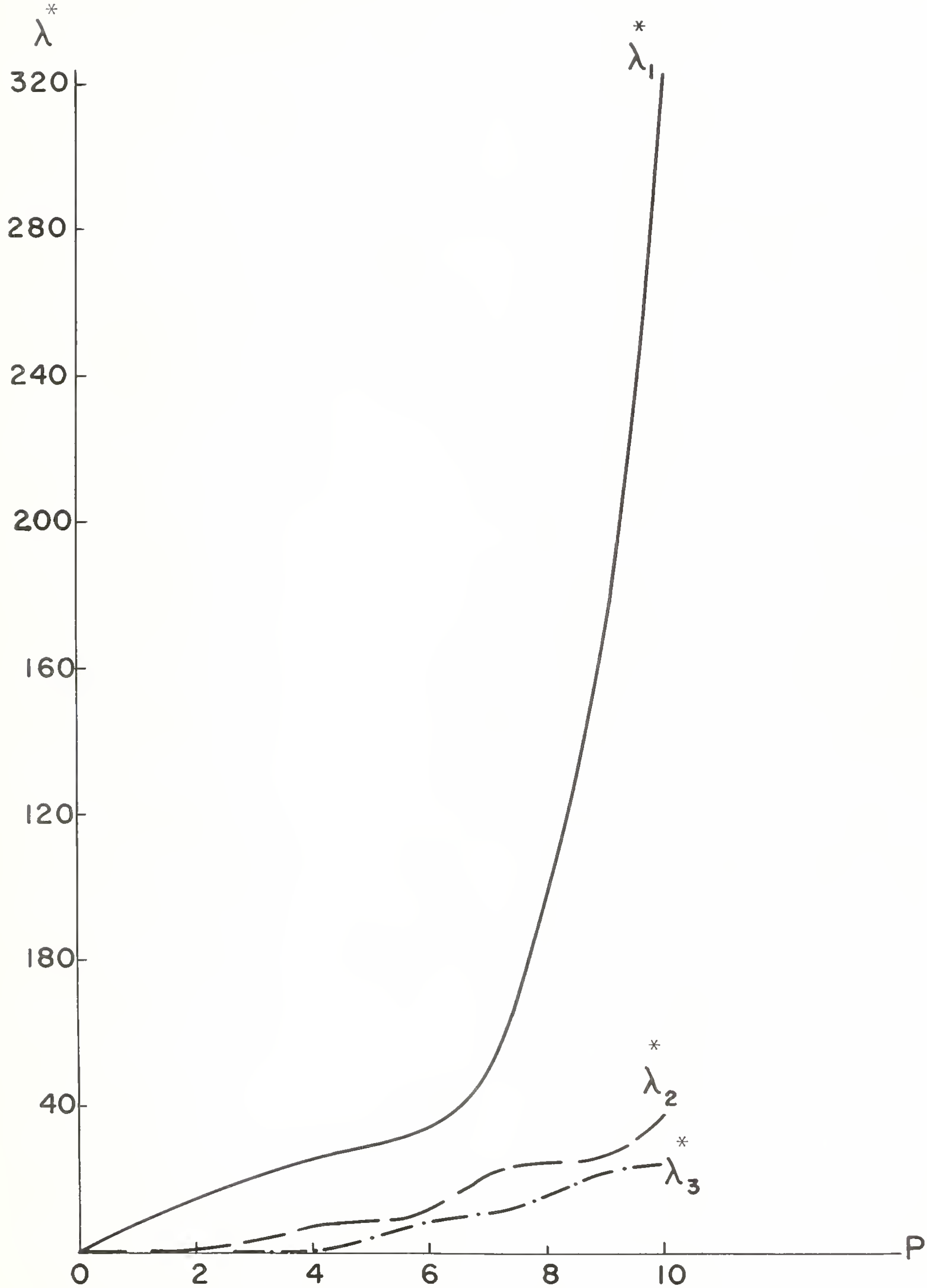
λ^*: Second Approximation



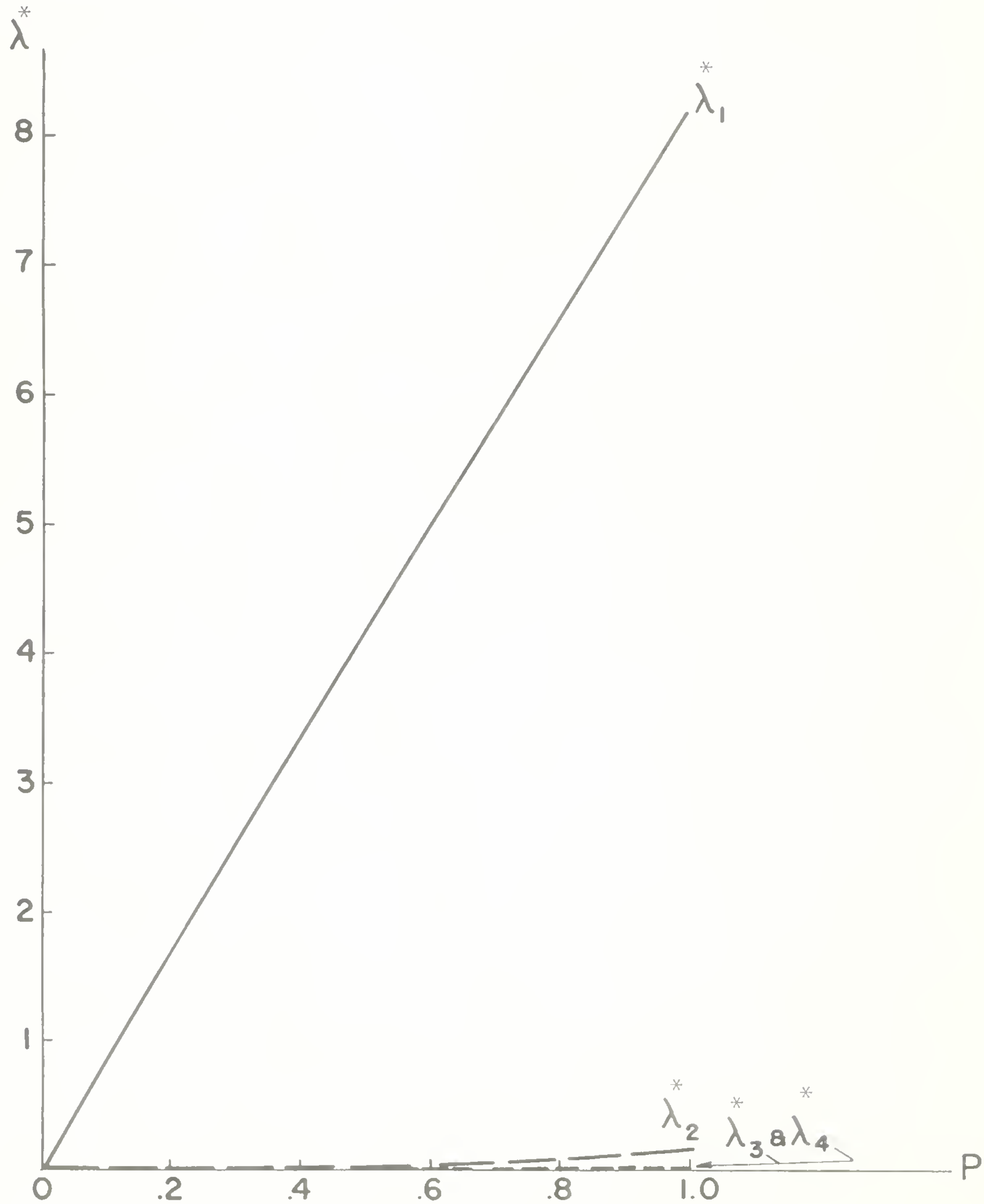
$\overset{*}{\lambda}$: Second Approximation



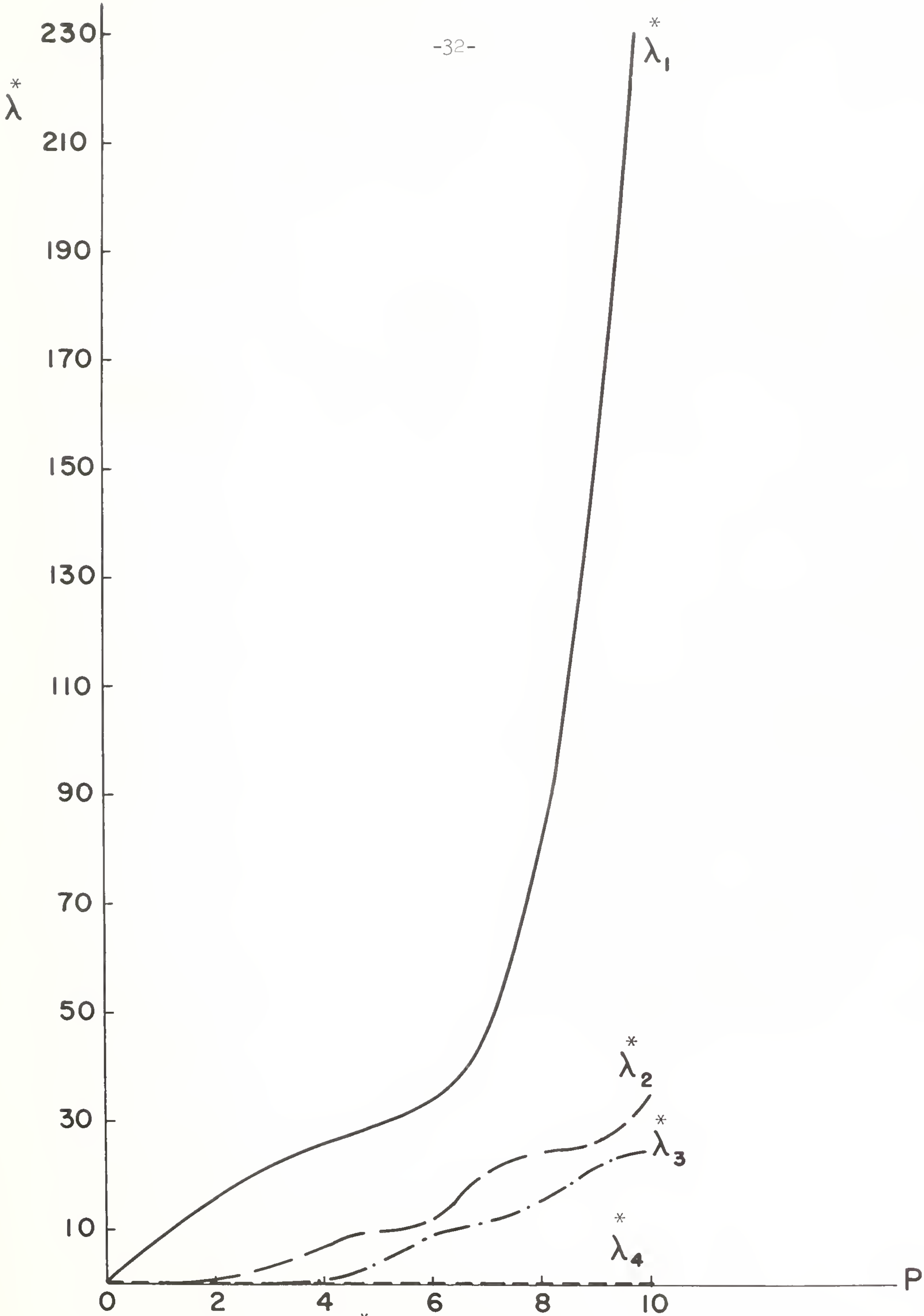
λ^* : Third Approximation



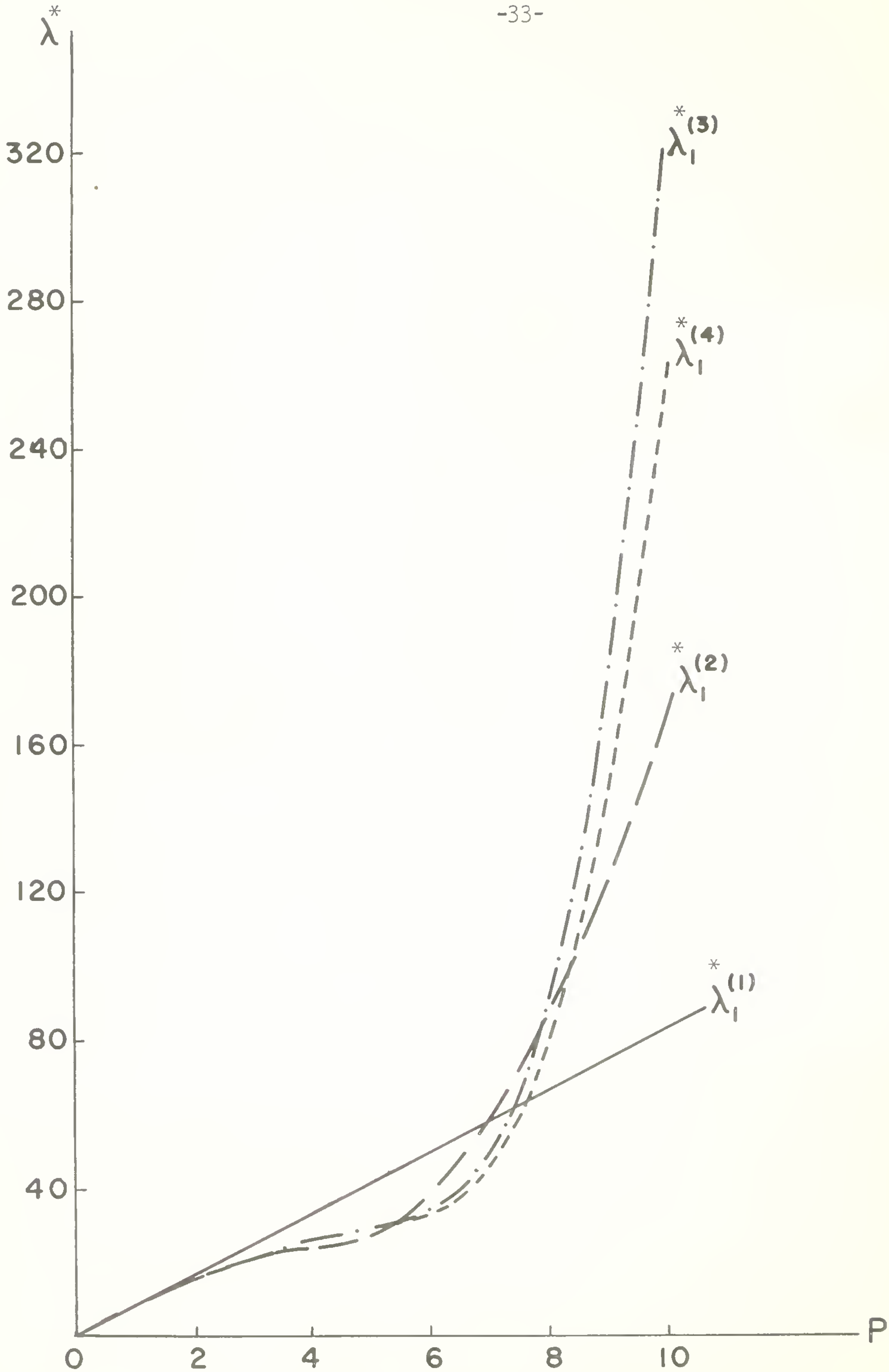
λ^* : Third Approximation



λ^* : Fourth Approximation



λ^*: Fourth Approximation



Graph of the Maximum λ^* in Each Approximation